

# On the Existence and Continuity of Equilibria for Two-Person Zero-Sum Games with Uncertain Payoffs

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## Abstract

This paper provides sufficient conditions for the existence of values and solutions for two-person zero-sum one-step games with infinite and possibly noncompact action sets for both players, possibly unbounded payoff functions, which may be neither convex nor concave. For such games, payoffs may not be defined for some pairs of strategies. In addition, the paper investigates continuity properties of the value functions and solution multifunctions, when action sets and payoffs depend on a parameter.

## 1 Introduction

This paper studies two-person zero-sum one-step games with possibly unbounded payoff functions, which may be neither convex nor concave, and with possibly noncompact action sets for both players. For such problems, payoffs may not be defined for some pairs of strategies because the corresponding expectations may not be well-defined. In addition, the standard minimax and infsup equalities and inequalities cannot be written because the both sides may be uncertain. In this paper, we introduce the values and solutions for such games, prove sufficient conditions for their existence, and establish continuity properties of the values and solution multifunctions if action sets and payoffs depend on a parameter.

As follows from the Nash equilibrium approach, the values and solutions may exist in spite of the fact that some outcomes can be uncertain. Let us consider a game with action sets  $\mathbb{A}$  and  $\mathbb{B}$  for Players I and II respectively and with a payoff function  $c : \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{R}$ , where  $c(a, b)$  is the payoff from Player I to Player II. We assume that  $\mathbb{A}$  and  $\mathbb{B}$  are standard Borel spaces and  $c(a, b)$  is a Borel function. Since some of the results of the paper deal with continuity properties of payoff functions and solutions, we assume without loss of generality that  $\mathbb{A}$  and  $\mathbb{B}$  are Borel subsets of Polish (complete, separable metric) spaces.

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The sets of strategies for Players I and II are the sets  $\mathbb{P}(\mathbb{A})$  and  $\mathbb{P}(\mathbb{B})$  of probability measures on  $\mathbb{A}$  and  $\mathbb{B}$  respectively. If Players I and II choose strategies  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  and  $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$  respectively, then Player I pays Player II the amount

$$\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) = \int_{\mathbb{A}} \int_{\mathbb{B}} c(a, b) \pi^{\mathbb{B}}(b) \pi^{\mathbb{A}}(da),$$

which may not be defined since the function  $c$  can be bounded neither below nor above. Therefore, the standard lower and upper values of the game

$$v_* = \sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \quad \text{and} \quad v^* = \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$$

cannot be defined. We consider natural conditions on the payoff function  $c$ , under which the payoffs are defined when either  $\pi^{\mathbb{A}}$  or  $\pi^{\mathbb{B}}$  is a pure strategy (in other words, when at least one player acts deterministically). In this case, it is possible to define

$$v_* = \sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \inf_{a \in \mathbb{A}} \hat{c}(a, \pi^{\mathbb{B}}) \quad \text{and} \quad v^* = \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \sup_{b \in \mathbb{B}} \hat{c}(\pi^{\mathbb{A}}, b), \quad (1.1)$$

where with a slight abuse of notations, we write  $\hat{c}(a, \pi^{\mathbb{B}})$  and  $\hat{c}(\pi^{\mathbb{A}}, b)$  instead of  $\hat{c}(\delta_{\{a\}}, \pi^{\mathbb{B}})$  and  $\hat{c}(\pi^{\mathbb{A}}, \delta_{\{b\}})$ , where  $\delta_{\{a\}}$  and  $\delta_{\{b\}}$  are Dirac measures concentrated at  $a \in \mathbb{A}$  and at  $b \in \mathbb{B}$  respectively.

The concept of solutions and the structure of the corresponding infsup equations becomes clear from the definition of a Nash equilibrium for this game. A pair of strategies  $(\pi_*^{\mathbb{A}}, \pi_*^{\mathbb{B}})$  of Players I and II form a Nash equilibrium, if  $\hat{c}(\pi_*^{\mathbb{A}}, \pi^{\mathbb{B}})$  and  $\hat{c}(\pi^{\mathbb{A}}, \pi_*^{\mathbb{B}})$  are well defined for all  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ ,  $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$  and

$$\hat{c}(\pi_*^{\mathbb{A}}, \pi^{\mathbb{B}}) \leq \hat{c}(\pi_*^{\mathbb{A}}, \pi_*^{\mathbb{B}}) \leq \hat{c}(\pi^{\mathbb{A}}, \pi_*^{\mathbb{B}}). \quad (1.2)$$

We denote by  $\mathbb{P}^S(\mathbb{A}) \subseteq \mathbb{P}(\mathbb{A})$  (and by  $\mathbb{P}^S(\mathbb{B}) \subseteq \mathbb{P}(\mathbb{B})$ ) the sets of strategies such that  $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$  is well defined for all  $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$  if  $\pi^{\mathbb{A}} \in \mathbb{P}^S(\mathbb{A})$  ( $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$  is well defined for all  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  if  $\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})$ ). The strategies from these sets are “safe” in the sense that the payoff can be calculated by the players playing these policies for each response of the opponent. In particular, pure strategies are safe under natural assumptions.

Thus, a pair of policies  $(\pi_*^{\mathbb{A}}, \pi_*^{\mathbb{B}})$  can form the Nash equilibrium even if there are policies  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  and  $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$  such that the integrals  $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$  are not defined. The lower and upper values corresponding to the Nash equilibrium solutions for this zero-sum game are

$$v^b = \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})} \inf_{\pi^{\mathbb{A}} \in \mathbb{P}^S(\mathbb{A})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \quad \text{and} \quad v^\sharp = \inf_{\pi^{\mathbb{A}} \in \mathbb{P}^S(\mathbb{A})} \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}).$$

Section 2 of this paper introduces the basic definitions and establishes some relations. In addition to the lower and upper values of the game, we also introduce weak lower values and weak upper values of the game. Section 3 studies the infsup equalities for the situation, when the action set for one of the players can be arbitrary, and establishes the lopsided minimax theorem (Theorem 3.3). In particular, Theorem 3.3 assumes neither convexity properties of payoffs nor compactness of the action set for one of the players. Appendix A provides comparisons of this theorem with several classic results. In particular, the lopsided minimax theorem in Aubin and Ekeland [2, Theorem 6.2.7] assumes convexity properties of payoffs, and Mertens et al. [17, Propositions I.1.9, I.2.2] assume compactness of the action set for one of the players. Section 4 provides sufficient conditions under which the uncertainty can be “neutralized”. Under these

conditions, the opponent can be punished by infinite losses, if she or he plays a policy, which is not safe. Example 5.5 demonstrates that the uncertainty neutralization may not take place even under strong continuity assumptions. Section 6 describes sufficient conditions for the existence of a weak value. Section 7 provides sufficient conditions for the existence of solutions for the game. Section 7 provides examples of games with and without solutions. Section 8 describes continuity properties of the values and solution multifunctions.

Our initial motivation for studying games with unbounded payoffs was originated by the progress in the theory of Markov Decision Processes with possibly noncompact action sets and unbounded costs [5, 8, 10, 12], that led to the extension of Berge's maximum theorem to possibly noncompact action sets; see Feinberg et al. [7, 9], Feinberg and Kasyanov [6]. These results were applied in [11] to games with perfect information. The results for games with simultaneous moves are limited in [11] to the situation when one of the players has a compact action set. This paper studies more general models that have significant potential applications to stochastic games; see Jaśhkewicz and Nowak [14], Mertens et al. [17, Chapter VII], and references therein for the literature on stochastic games.

## 2 Definitions and Preliminary Results

Let  $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$  and  $\mathbb{S}$  be a metric space. For a nonempty set  $S \subseteq \mathbb{S}$ , the notation  $f : S \subseteq \mathbb{S} \rightarrow \overline{\mathbb{R}}$  means that for each  $s \in S$  the value  $f(s) \in \overline{\mathbb{R}}$  is defined. In general, the function  $f$  may be also defined outside of  $S$ . The notation  $f : \mathbb{S} \rightarrow \overline{\mathbb{R}}$  means that the function  $f$  is defined on the entire space  $\mathbb{S}$ . This notation is equivalent to the notation  $f : S \subseteq \mathbb{S} \rightarrow \overline{\mathbb{R}}$ , which we do not write explicitly. For a function  $f : S \subseteq \mathbb{S} \rightarrow \overline{\mathbb{R}}$  we sometimes consider its restriction  $f|_{\tilde{S}} : \tilde{S} \subseteq S \rightarrow \overline{\mathbb{R}}$  to the set  $\tilde{S} \subseteq S$ . Throughout the paper we denote by  $\mathbb{K}(\mathbb{S})$  the family of all nonempty compact subsets of  $\mathbb{S}$  and by  $S(\mathbb{S})$  the family of all nonempty subsets of  $\mathbb{S}$ .

We recall that, for a nonempty set  $S \subseteq \mathbb{S}$ , a function  $f : S \subseteq \mathbb{S} \rightarrow \overline{\mathbb{R}}$  is called *lower semi-continuous at  $s \in S$* , if for each sequence  $\{s^{(n)}\}_{n=1,2,\dots} \subset S$ , that converges to  $s$  in  $\mathbb{S}$ , the inequality  $\liminf_{n \rightarrow \infty} f(s^{(n)}) \geq f(s)$  holds. A function  $f : S \subseteq \mathbb{S} \rightarrow \overline{\mathbb{R}}$  is called *upper semi-continuous at  $s \in S$* , if  $-f$  is lower semi-continuous at  $s \in S$ . Consider the level sets

$$\mathcal{D}_f(\lambda; S) := \{s \in S : f(s) \leq \lambda\}, \quad \lambda \in \mathbb{R}.$$

The level sets  $\mathcal{D}_f(\lambda; S)$  satisfy the following properties that are used in this paper:

- (a) if  $\lambda_1 > \lambda$ , then  $\mathcal{D}_f(\lambda; S) \subseteq \mathcal{D}_f(\lambda_1; S)$ ;
- (b) if  $g, f$  are functions on  $S$  satisfying  $g(s) \geq f(s)$  for all  $s \in S$  then  $\mathcal{D}_g(\lambda; S) \subseteq \mathcal{D}_f(\lambda; S)$ .

A function  $f : S \subseteq \mathbb{S} \rightarrow \overline{\mathbb{R}}$  is called *lower / upper semi-continuous*, if  $f$  is lower / upper semi-continuous at each  $s \in S$ . A function  $f : S \subseteq \mathbb{S} \rightarrow \overline{\mathbb{R}}$  is called *inf-compact on  $S$* , if all the level sets  $\{\mathcal{D}_f(\lambda; S)\}_{\lambda \in \mathbb{R}}$  are compact in  $\mathbb{S}$ . A function  $f : S \subseteq \mathbb{S} \rightarrow \overline{\mathbb{R}}$  is called *sup-compact on  $S$* , if  $-f$  is inf-compact on  $S$ .

Each nonempty subset  $S$  of a metric space  $\mathbb{S}$  can be considered as a metric space with the same metric. Let  $\mathbb{K}(S)$  be the family of all compact subsets of the metric space  $S$ .

**Remark 2.1.** For each nonempty subset  $S \subseteq \mathbb{S}$  the following equality holds:

$$\mathbb{K}(S) = \{C \subseteq S : C \in \mathbb{K}(\mathbb{S})\}.$$

**Remark 2.2.** It is well-known that a function  $f : \mathbb{S} \rightarrow \overline{\mathbb{R}}$  is lower semi-continuous if and only if the set  $\mathcal{D}_f(\lambda; \mathbb{S})$  is closed for every  $\lambda \in \mathbb{R}$ ; see e.g., Aubin [1, p. 12, Proposition 1.4]. For a function  $f : S \subseteq \mathbb{S} \rightarrow \overline{\mathbb{R}}$ , let  $\tilde{f}$  be the function  $\tilde{f} : \mathbb{S} \rightarrow \overline{\mathbb{R}}$ , defined as  $\tilde{f}(s) := f(s)$ , when  $s \in S$ , and  $\tilde{f}(s) := +\infty$  otherwise. Then the function  $\tilde{f} : \mathbb{S} \rightarrow \overline{\mathbb{R}}$  is lower semi-continuous if and only if for each  $\lambda \in \mathbb{R}$  the set  $\mathcal{D}_{\tilde{f}}(\lambda; \mathbb{S})$  is closed in  $\mathbb{S}$ .

For a metric space  $\mathbb{S}$ , let  $\mathcal{B}(\mathbb{S})$  be a Borel  $\sigma$ -field on  $\mathbb{S}$ , that is, the  $\sigma$ -field generated by all open sets of the metric space  $\mathbb{S}$ . For a nonempty Borel subset  $S \subset \mathbb{S}$ , denote by  $\mathcal{B}(S)$  the  $\sigma$ -field whose elements are intersections of  $S$  with elements of  $\mathcal{B}(\mathbb{S})$ . Observe that  $S$  is a metric space with the same metric as on  $\mathbb{S}$ , and  $\mathcal{B}(S)$  is its Borel  $\sigma$ -field. For a metric space  $\mathbb{S}$ , let  $\mathbb{P}(\mathbb{S})$  be the set of probability measures on  $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$  and  $\mathbb{P}^{fs}(\mathbb{S})$  denotes the set of all probability measures whose supports are finite subsets of  $\mathbb{S}$ . A sequence of probability measures  $\{\mu^{(n)}\}_{n=1,2,\dots}$  from  $\mathbb{P}(\mathbb{S})$  converges weakly to  $\mu \in \mathbb{P}(\mathbb{S})$  if for each bounded continuous function  $f$  on  $\mathbb{S}$

$$\int_{\mathbb{S}} f(s) \mu^{(n)}(ds) \rightarrow \int_{\mathbb{S}} f(s) \mu(ds) \quad \text{as } n \rightarrow \infty.$$

We endow  $\mathbb{P}(\mathbb{S})$  with the topology of weak convergence of probability measures on  $\mathbb{S}$ . If  $\mathbb{S}$  is a separable metric space, then  $\mathbb{P}(\mathbb{S})$  is separable metric space too; Parthasarathy [18, Chapter II, Theorem 6.3]. Moreover, the set  $\mathbb{P}^{fs}(\mathbb{S})$  is dense in  $\mathbb{P}(\mathbb{S})$ .

An integral  $\int_{\mathbb{S}} f(s) \mu(ds)$  of a measurable  $\overline{\mathbb{R}}$ -valued function  $f$  on  $\mathbb{S}$  over the measure  $\mu \in \mathbb{P}(\mathbb{S})$  is well-defined if either  $\int_{\mathbb{S}} f^-(s) \mu(ds) > -\infty$  or  $\int_{\mathbb{S}} f^+(s) \mu(ds) < +\infty$ , where  $f^-(s) = \min\{f(s), 0\}$ ,  $f^+(s) = \max\{f(s), 0\}$ ,  $s \in \mathbb{S}$ . If the integral is well-defined, then  $\int_{\mathbb{S}} f(s) \mu(ds) := \int_{\mathbb{S}} f^+(s) \mu(ds) + \int_{\mathbb{S}} f^-(s) \mu(ds)$ .

**Definition 2.3.** A two-person zero-sum game is a triplet  $\{\mathbb{A}, \mathbb{B}, c\}$ , where

- (i)  $\mathbb{A}$  is the space of actions for Player I, which is a nonempty Borel subset of a Polish space;
- (ii)  $\mathbb{B}$  is the space of actions for Player II, which is a nonempty Borel subset of a Polish space;
- (iii) the payoff from Player I to Player II,  $-\infty < c(a, b) < +\infty$ , for choosing actions  $a \in \mathbb{A}$  and  $b \in \mathbb{B}$ , is a measurable function on  $\mathbb{A} \times \mathbb{B}$ ;
- (iv) for each  $b \in \mathbb{B}$  the function  $a \rightarrow c(a, b)$  is bounded from below on  $\mathbb{A}$ ;
- (v) for each  $a \in \mathbb{A}$  the function  $b \rightarrow c(a, b)$  is bounded from above on  $\mathbb{B}$ .

The game is played as follows:

- a decision-makers (Players I and II) choose simultaneously respective actions  $a \in \mathbb{A}$  and  $b \in \mathbb{B}$ ;
- the result  $(a, b)$  is announced to both of them;
- Player I pays Player II the amount  $c(a, b)$ .

Mixed strategies for Players I and II are probability measures  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  and  $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$ . Moreover,  $\pi^{\mathbb{A}}$  ( $\pi^{\mathbb{B}}$ ) is called pure, if the probability measure  $\pi^{\mathbb{A}}(\cdot)$  ( $\pi^{\mathbb{B}}(\cdot)$ ) is concentrated at one point. Note that  $\mathbb{P}(\mathbb{A})$  is the set of mixed strategies for Player I, and  $\mathbb{P}(\mathbb{B})$  is the set of mixed strategies for Player II.

**Remark 2.4.** If a triplet  $\{\mathbb{A}, \mathbb{B}, c\}$  is the two-person zero-sum game defined above, then the triplet  $\{\mathbb{B}, \mathbb{A}, -c^{\mathbb{A} \leftrightarrow \mathbb{B}}\}$ , where  $c^{\mathbb{A} \leftrightarrow \mathbb{B}}(b, a) := c(a, b)$  for each  $a \in \mathbb{A}$  and  $b \in \mathbb{B}$ , is also a game satisfying conditions (i–v) from Definition 2.3.

**Remark 2.5.** Assumptions (iv) and (v) for the game  $\{\mathbb{A}, \mathbb{B}, c\}$  are natural because without them the expected payoffs are possibly undefined even if the one of the players chooses a pure strategy.

Let us set

$$c^+(a, b) := \max\{c(a, b), 0\}, \quad c^-(a, b) := \min\{c(a, b), 0\},$$

for each  $(a, b) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$ , and

$$\hat{c}^+(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) := \int_{\mathbb{A}} \int_{\mathbb{B}} c^+(a, b) \pi^{\mathbb{B}}(db) \pi^{\mathbb{A}}(da), \quad \hat{c}^-(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) := \int_{\mathbb{A}} \int_{\mathbb{B}} c^-(a, b) \pi^{\mathbb{B}}(db) \pi^{\mathbb{A}}(da),$$

for each  $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$ . Then the *expected payoff* from Player I to Player II,

$$\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) := \hat{c}^+(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) + \hat{c}^-(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}),$$

is well-defined if either  $\hat{c}^+(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) < +\infty$  or  $\hat{c}^-(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) > -\infty$ , where  $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$ . Of course, when the function  $c$  is unbounded both below as well as above, the quantity  $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$  is possibly *undefined* for some  $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$ . We denote:

$$\begin{aligned} \mathbb{P}_{\pi^{\mathbb{A}}}^S(\mathbb{B}) &:= \{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B}) : \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \text{ is well-defined}\}, & \pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A}); \\ \mathbb{P}_{\pi^{\mathbb{B}}}^S(\mathbb{A}) &:= \{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A}) : \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \text{ is well-defined}\}, & \pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B}). \end{aligned}$$

Further, with a slight abuse of notations, we write  $\hat{c}(a, \pi^{\mathbb{B}})$  and  $\hat{c}(\pi^{\mathbb{A}}, b)$  instead of  $\hat{c}(\delta_{\{a\}}, \pi^{\mathbb{B}})$  and  $\hat{c}(\pi^{\mathbb{A}}, \delta_{\{b\}})$  for each  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  and  $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$ , where  $\delta_{\{a\}}$  and  $\delta_{\{b\}}$  are Dirac measures concentrated at  $a \in \mathbb{A}$  and at  $b \in \mathbb{B}$  respectively.

**Remark 2.6.** Assumption (iv) for the game  $\{\mathbb{A}, \mathbb{B}, c\}$  yields that  $\hat{c}^-(\pi^{\mathbb{A}}, b) > -\infty$  for each  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  and  $b \in \mathbb{B}$ . Therefore,  $\mathbb{P}^{fs}(\mathbb{B}) \subset \mathbb{P}_{\pi^{\mathbb{A}}}^S(\mathbb{B})$  for each  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  and, since  $\mathbb{P}^{fs}(\mathbb{B})$  is dense in  $\mathbb{P}(\mathbb{B})$ , then  $\bigcap_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \mathbb{P}_{\pi^{\mathbb{A}}}^S(\mathbb{B})$  is dense in  $\mathbb{P}(\mathbb{B})$ .

**Remark 2.7.** Assumption (v) for the game  $\{\mathbb{A}, \mathbb{B}, c\}$  implies that  $\hat{c}^+(a, \pi^{\mathbb{B}}) < +\infty$  for each  $a \in \mathbb{A}$  and  $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$ . Thus,  $\mathbb{P}^{fs}(\mathbb{A}) \subset \mathbb{P}_{\pi^{\mathbb{B}}}^S(\mathbb{A})$  for each  $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$  and, since  $\mathbb{P}^{fs}(\mathbb{A})$  is dense in  $\mathbb{P}(\mathbb{A})$ , then  $\bigcap_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \mathbb{P}_{\pi^{\mathbb{B}}}^S(\mathbb{A})$  is dense in  $\mathbb{P}(\mathbb{A})$ .

The set of mixed strategies for each player is partitioned into the sets of safe strategies  $\mathbb{P}^S(\mathbb{A})$  and  $\mathbb{P}^S(\mathbb{B})$  (strategies, for which the expected payoff is well-defined for all strategies played by another player) and unsafe strategies  $\mathbb{P}^U(\mathbb{A})$  and  $\mathbb{P}^U(\mathbb{B})$ :

$$\begin{aligned} \mathbb{P}^S(\mathbb{A}) &:= \{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A}) : \mathbb{P}_{\pi^{\mathbb{A}}}^S(\mathbb{B}) = \mathbb{P}(\mathbb{B})\}, & \mathbb{P}^U(\mathbb{A}) &:= \{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A}) : \mathbb{P}_{\pi^{\mathbb{A}}}^S(\mathbb{B}) \neq \mathbb{P}(\mathbb{B})\}; \\ \mathbb{P}^S(\mathbb{B}) &:= \{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B}) : \mathbb{P}_{\pi^{\mathbb{B}}}^S(\mathbb{A}) = \mathbb{P}(\mathbb{A})\}, & \mathbb{P}^U(\mathbb{B}) &:= \{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B}) : \mathbb{P}_{\pi^{\mathbb{B}}}^S(\mathbb{A}) \neq \mathbb{P}(\mathbb{A})\}. \end{aligned}$$

**Remark 2.8.** We note that  $\mathbb{P}(\mathbb{A}) = \mathbb{P}^S(\mathbb{A}) \cup \mathbb{P}^U(\mathbb{A})$ ,  $\mathbb{P}(\mathbb{B}) = \mathbb{P}^S(\mathbb{B}) \cup \mathbb{P}^U(\mathbb{B})$ ,  $\mathbb{P}^S(\mathbb{A}) \cap \mathbb{P}^U(\mathbb{A}) = \emptyset$ , and  $\mathbb{P}^S(\mathbb{B}) \cap \mathbb{P}^U(\mathbb{B}) = \emptyset$ . Moreover,  $\mathbb{P}^{fs}(\mathbb{A}) \subset \mathbb{P}^S(\mathbb{A})$  (see Assumption (iv) from Definition 2.3 and Remark 2.6) and  $\mathbb{P}^{fs}(\mathbb{B}) \subset \mathbb{P}^S(\mathbb{B})$  (see Assumption (v) from Definition 2.3 and Remark 2.7). Therefore,  $\mathbb{P}^S(\mathbb{A})$  is dense in  $\mathbb{P}(\mathbb{A})$  and  $\mathbb{P}^S(\mathbb{B})$  is dense in  $\mathbb{P}(\mathbb{B})$ .

Let us introduce the following notations:

$$\begin{aligned} \hat{c}^\#(\pi^\mathbb{A}) &:= \sup_{b \in \mathbb{B}} \hat{c}(\pi^\mathbb{A}, b), \quad \mathbb{P}_\alpha^\#(\mathbb{A}) := \{\pi_*^\mathbb{A} \in \mathbb{P}(\mathbb{A}) : \hat{c}^\#(\pi_*^\mathbb{A}) \leq \alpha\}, \quad \pi^\mathbb{A} \in \mathbb{P}(\mathbb{A}), \alpha \in \overline{\mathbb{R}}, \\ \hat{c}^\flat(\pi^\mathbb{B}) &:= \inf_{a \in \mathbb{A}} \hat{c}(a, \pi^\mathbb{B}), \quad \mathbb{P}_\beta^\flat(\mathbb{B}) := \{\pi_*^\mathbb{B} \in \mathbb{P}(\mathbb{B}) : \hat{c}^\flat(\pi_*^\mathbb{B}) \geq \beta\}, \quad \pi^\mathbb{B} \in \mathbb{P}(\mathbb{B}), \beta \in \overline{\mathbb{R}}. \end{aligned} \quad (2.1)$$

Remarks 2.6 and 2.7 imply respectively that  $\hat{c}^\#(\pi^\mathbb{A}) > -\infty$  for all  $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$  and  $\hat{c}^\flat(\pi^\mathbb{B}) < +\infty$  for all  $\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})$ .

**Theorem 2.9.** (Feinberg et al. [11, Theorem 5.7]) *Let  $\{\mathbb{A}, \mathbb{B}, c\}$  be a two-person zero-sum game introduced in Definition 2.3 and  $(\pi^\mathbb{A}, \pi^\mathbb{B}) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$ . Then the following two equalities hold:*

$$\hat{c}^\#(\pi^\mathbb{A}) = \sup_{\pi_*^\mathbb{B} \in \mathbb{P}_{\pi^\mathbb{A}}^S(\mathbb{B})} \hat{c}(\pi^\mathbb{A}, \pi_*^\mathbb{B}); \quad (2.2)$$

$$\hat{c}^\flat(\pi^\mathbb{B}) = \inf_{\pi_*^\mathbb{A} \in \mathbb{P}_{\pi^\mathbb{B}}^S(\mathbb{A})} \hat{c}(\pi_*^\mathbb{A}, \pi^\mathbb{B}), \quad (2.3)$$

where  $\hat{c}^\#$  and  $\hat{c}^\flat$  are defined in (2.1).

**Remark 2.10.** According to (2.1) and Assumptions (iv) and (v) from Definition 2.3 for the game  $\{\mathbb{A}, \mathbb{B}, c\}$  (see also Remarks 2.6 and 2.7 and Theorem 2.9), the inequality

$$\hat{c}^\flat(\pi^\mathbb{B}) \leq \hat{c}^\#(\pi^\mathbb{A}) \quad (2.4)$$

holds for each  $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$  and  $\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})$  such that  $\hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B})$  is well-defined, that is, inequality (2.4) holds, if  $\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})$  and  $\pi^\mathbb{A} \in \mathbb{P}_{\pi^\mathbb{B}}^S(\mathbb{A})$ , or equivalently, if  $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$  and  $\pi^\mathbb{B} \in \mathbb{P}_{\pi^\mathbb{A}}^S(\mathbb{B})$ . Indeed, for  $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$  and  $\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})$  such that  $\hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B})$  is well-defined, the following relations hold:

$$\hat{c}^\flat(\pi^\mathbb{B}) = \inf_{\pi_*^\mathbb{A} \in \mathbb{P}_{\pi^\mathbb{B}}^S(\mathbb{A})} \hat{c}(\pi_*^\mathbb{A}, \pi^\mathbb{B}) \leq \hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B}) \leq \sup_{\pi_*^\mathbb{B} \in \mathbb{P}_{\pi^\mathbb{A}}^S(\mathbb{B})} \hat{c}(\pi^\mathbb{A}, \pi_*^\mathbb{B}) = \hat{c}^\#(\pi^\mathbb{A}).$$

Since it is not clear whether inequality (2.4) holds for  $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$  and  $\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})$  with  $\hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B})$  is undefined, the following definition introduces the lower and upper value (conservative values of each Player) in the slightly more “accurate” formulations than it is usually done.

**Definition 2.11.** *The following quantities*

$$v^\flat := \sup_{\pi^\mathbb{B} \in \mathbb{P}^S(\mathbb{B})} \hat{c}^\flat(\pi^\mathbb{B}) \quad \text{and} \quad v^\# := \inf_{\pi^\mathbb{A} \in \mathbb{P}^S(\mathbb{A})} \hat{c}^\#(\pi^\mathbb{A}) \quad (2.5)$$

define the lower and upper values of the game  $\{\mathbb{A}, \mathbb{B}, c\}$ , and

$$v_w^\flat := \inf_{\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})} \sup_{\pi^\mathbb{B} \in \mathbb{P}_{\pi^\mathbb{A}}^S(\mathbb{B})} \hat{c}^\flat(\pi^\mathbb{B}) \quad \text{and} \quad v_w^\# := \sup_{\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})} \inf_{\pi^\mathbb{A} \in \mathbb{P}_{\pi^\mathbb{B}}^S(\mathbb{A})} \hat{c}^\#(\pi^\mathbb{A}) \quad (2.6)$$

define the lower and upper weak values of the game  $\{\mathbb{A}, \mathbb{B}, c\}$ .



**Remark 2.12.** Theorem 2.9 and equality (2.5) yield that

$$v^b := \sup_{\pi^B \in \mathbb{P}^S(\mathbb{B})} \inf_{\pi^A \in \Delta(\mathbb{A})} \hat{c}(\pi^A, \pi^B) \quad \text{and} \quad v^\sharp = \inf_{\pi^A \in \mathbb{P}^S(\mathbb{A})} \sup_{\pi^B \in \Delta(\mathbb{B})} \hat{c}(\pi^A, \pi^B),$$

for each  $\Delta(\mathbb{A}) \subset \mathbb{P}(\mathbb{A})$  and  $\Delta(\mathbb{B}) \subset \mathbb{P}(\mathbb{B})$  such that  $\Delta(\mathbb{A})$  and  $\Delta(\mathbb{B})$  contain all pure strategies for Player I and II respectively. In particular,

$$v^b := \sup_{\pi^B \in \mathbb{P}^S(\mathbb{B})} \inf_{\pi^A \in \mathbb{P}^S(\mathbb{A})} \hat{c}(\pi^A, \pi^B) \quad \text{and} \quad v^\sharp = \inf_{\pi^A \in \mathbb{P}^S(\mathbb{A})} \sup_{\pi^B \in \mathbb{P}^S(\mathbb{B})} \hat{c}(\pi^A, \pi^B). \quad (2.7)$$

Similarly, Theorem 2.9 and equality (2.6) imply that

$$v_w^b = \inf_{\pi^A \in \mathbb{P}(\mathbb{A})} \sup_{\pi^B \in \mathbb{P}_{\pi^A}^S(\mathbb{B})} \inf_{\pi_*^A \in \mathbb{P}_{\pi^B}^S(\mathbb{A})} \hat{c}(\pi_*^A, \pi^B) \quad \text{and} \quad v_w^\sharp = \sup_{\pi^B \in \mathbb{P}(\mathbb{B})} \inf_{\pi^A \in \mathbb{P}_{\pi^B}^S(\mathbb{A})} \sup_{\pi_*^A \in \mathbb{P}_{\pi^B}^S(\mathbb{A})} \hat{c}(\pi^A, \pi_*^A).$$

**Remark 2.13.** Let  $\{\mathbb{A}, \mathbb{B}, c\}$  be a two-person zero-sum game introduced in Definition 2.3. Then

$$v^b \leq v_w^b \leq v_w^\sharp \leq v^\sharp. \quad (2.8)$$

Indeed,

$$v^b \leq v_w^b, \quad (2.9)$$

because  $\mathbb{P}^S(\mathbb{B}) \subset \mathbb{P}_{\pi^A}^S(\mathbb{B})$  for each  $\pi^A \in \mathbb{P}(\mathbb{A})$ . Moreover,

$$v_w^b = \inf_{\pi^A \in \mathbb{P}(\mathbb{A})} \sup_{\pi^B \in \mathbb{P}_{\pi^A}^S(\mathbb{B})} \hat{c}^b(\pi^B) \leq \sup_{\pi^B \in \mathbb{P}(\mathbb{B})} \hat{c}^b(\pi^B) \leq \sup_{\pi^B \in \mathbb{P}(\mathbb{B})} \inf_{\pi^A \in \mathbb{P}_{\pi^B}^S(\mathbb{A})} \hat{c}^\sharp(\pi^A) = v_w^\sharp, \quad (2.10)$$

where the equalities follow from (2.6); the first inequality holds because  $\mathbb{P}_{\pi^A}^S(\mathbb{B}) = \mathbb{P}(\mathbb{B})$  for each  $\pi^A \in \mathbb{P}^{fs}(\mathbb{A})$  (see Remark 2.8); and the second inequality follows from inequality (2.4) for  $\pi^B \in \mathbb{P}(\mathbb{B})$  and  $\pi^A \in \mathbb{P}_{\pi^B}^S(\mathbb{A})$  (see Remark 2.10). Note that the inequality  $v_w^\sharp \leq v^\sharp$  follows from inequality (2.9) and the symmetric reasonings.

We also remark that inequalities (2.10), being applied to the game  $\{\mathbb{B}, \mathbb{A}, -c^{\mathbb{A} \leftrightarrow \mathbb{B}}\}$  (see Remark 2.4), yield that

$$v_w^b \leq \inf_{\pi^A \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^A) \leq v_w^\sharp. \quad (2.11)$$

and these inequalities are symmetric to (2.10). However, it is currently unknown whether without additional assumptions

$$\sup_{\pi^B \in \mathbb{P}(\mathbb{B})} \hat{c}^b(\pi^B) \leq \inf_{\pi^A \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^A).$$

**Remark 2.14.** Generally speaking, we can not claim that the functions  $\pi^A \rightarrow \sup_{\pi_*^B \in \mathbb{P}_{\pi^A}^S(\mathbb{B})} \hat{c}^b(\pi_*^B)$  and  $\pi^B \rightarrow \inf_{\pi_*^A \in \mathbb{P}_{\pi^B}^S(\mathbb{A})} \hat{c}^\sharp(\pi_*^A)$  are constant on  $\mathbb{P}(\mathbb{A})$  and  $\mathbb{P}(\mathbb{B})$  respectively. If these functions are constant, then inequalities (2.8) yield that

$$v^b \leq v_w^b = \sup_{\pi^B \in \mathbb{P}(\mathbb{B})} \hat{c}^b(\pi^B) \leq \inf_{\pi^A \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^A) = v_w^\sharp \leq v^\sharp.$$

**Remark 2.15.** The lower weak value  $v_w^b$  of the game  $\{\mathbb{A}, \mathbb{B}, c\}$  is the greatest lower bound of all naturally defined lower values for the game  $\{\mathbb{A}, \mathbb{B}, c\}$ . In particular,  $v_w^b \leq \sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}})$ . Similarly, the upper weak value  $v_w^\sharp$  of the game  $\{\mathbb{A}, \mathbb{B}, c\}$  is the least upper bound of all naturally defined upper values of the game  $\{\mathbb{A}, \mathbb{B}, c\}$ . In particular,  $\inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^{\mathbb{A}}) \leq v_w^\sharp$ . Theorem 5.1 below describes sufficient conditions for the equalities

$$v_w^b = \sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}}) = \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^{\mathbb{A}}) = v_w^\sharp.$$

**Remark 2.16.** If  $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$  is well-defined for each  $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$ , in particular, if  $c$  is bounded from below or above on  $\mathbb{A} \times \mathbb{B}$ , then the lower and upper values defined by (2.6) and (2.5) coincide with their classical definitions, that is,

$$\begin{aligned} v_w^b = v^b &= \sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}}) = \sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}), \\ v_w^\sharp = v^\sharp &= \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^{\mathbb{A}}) = \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}); \end{aligned}$$

see also Theorem 2.9.

According to Remarks 2.13, 2.15 and 2.16, a value and a weak value can be naturally defined as follows.

**Definition 2.17.** If the equality

$$v^b = v^\sharp \quad (v_w^b = v_w^\sharp) \tag{2.12}$$

holds, then we say that this common quantity is the value of the game  $\{\mathbb{A}, \mathbb{B}, c\}$  (the weak value of the game  $\{\mathbb{A}, \mathbb{B}, c\}$  respectively). We denote this value by  $v$  ( $v_w$  respectively).

**Remark 2.18.** If the value  $v$  of the game  $\{\mathbb{A}, \mathbb{B}, c\}$  exists, then there exists the weak value  $v_w$  of the game  $\{\mathbb{A}, \mathbb{B}, c\}$  and  $v = v_w$ ; see Remark 2.13.

**Remark 2.19.** Let  $\{\mathbb{A}, \mathbb{B}, c\}$  be a two-person zero-sum game introduced in Definition 2.3. If the weak value  $v_w$  of the game  $\{\mathbb{A}, \mathbb{B}, c\}$  exists, then the following equalities hold:

$$v_w = \sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}}) = \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^{\mathbb{A}}). \tag{2.13}$$

Indeed, formulae (2.10), being applied twice to the games  $\{\mathbb{A}, \mathbb{B}, c\}$  and  $\{\mathbb{B}, \mathbb{A}, -c^{\mathbb{A} \leftrightarrow \mathbb{B}}\}$  respectively (see Remark 2.4) yield the following inequalities:

$$v_w^b \leq \sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}}) \leq v_w^\sharp \quad \text{and} \quad -v_w^\sharp \leq -\inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^{\mathbb{A}}) \leq -v_w^b. \tag{2.14}$$

Since  $v_w^b = v_w^\sharp$ , then inequalities (2.14) yield equalities (2.13).

Let us set

$$\begin{aligned} \mathbb{P}_{<\alpha}^\sharp(\mathbb{A}) &:= \{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A}) : \hat{c}^\sharp(\pi^{\mathbb{A}}) < \alpha\}, \quad \alpha \in \overline{\mathbb{R}}, \\ \mathbb{P}_{>\beta}^b(\mathbb{B}) &:= \{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B}) : \hat{c}^b(\pi^{\mathbb{B}}) > \beta\}, \quad \beta \in \overline{\mathbb{R}}, \end{aligned}$$

where  $\hat{c}^\sharp$  and  $\hat{c}^b$  are defined in (2.1).



**Lemma 2.20.** (Feinberg et al. [11, Lemma 5.10]) *Let  $\{\mathbb{A}, \mathbb{B}, c\}$  be a two-person zero-sum game introduced in Definition 2.3. Then the following statements hold:*

- (a) *the function  $\hat{c}^\sharp$  is convex on  $\mathbb{P}(\mathbb{A})$ ;*
- (b) *the function  $\hat{c}^\flat$  is concave on  $\mathbb{P}(\mathbb{B})$ ;*
- (c) *the sets  $\mathbb{P}_\alpha^\sharp(\mathbb{A})$ ,  $\mathbb{P}_{<\alpha}^\sharp(\mathbb{A})$ ,  $\mathbb{P}_\beta^\flat(\mathbb{B})$ , and  $\mathbb{P}_{>\beta}^\flat(\mathbb{B})$  are convex for all  $\alpha, \beta \in \overline{\mathbb{R}}$ ;*

### 3 Lopsided Minimax Theorem

In this section we consider lopsided games defined as follows.

**Definition 3.1.** *A lopsided two-person zero-sum game is a triplet  $\{\mathbb{A}, \mathbb{B}, c\}$ , where*

- (i)  *$\mathbb{A}$  is the space of actions for Player I, which is a nonempty Borel subset of a Polish space;*
- (ii)  *$\mathbb{B}$  is the space of actions for Player II, which is a nonempty set;*
- (iii) *the payoff from Player I to Player II,  $-\infty < c(a, b) < +\infty$ , for choosing actions  $a \in \mathbb{A}$  and  $b \in \mathbb{B}$ , such that  $c(\cdot, b)$  is a measurable function on  $\mathbb{A}$  for each  $b \in \mathbb{B}$ ;*
- (iv) *for each  $b \in \mathbb{B}$  the function  $a \rightarrow c(a, b)$  is bounded from below on  $\mathbb{A}$ .*

**Remark 3.2.** If  $\{\mathbb{A}, \mathbb{B}, c\}$  and  $\{\mathbb{B}, \mathbb{A}, -c^{\mathbb{A} \leftrightarrow \mathbb{B}}\}$  are lopsided two-person zero-sum games introduced in Definition 3.1 (see also Remark 2.4) such that  $c$  is measurable on  $\mathbb{A} \times \mathbb{B}$ , then the both triples are two-person zero-sum games introduced in Definition 2.3.

Let  $\{\mathbb{A}, \mathbb{B}, c\}$  be a lopsided two-person zero-sum game introduced in Definition 3.1. We denote the set of all probability measures whose supports are finite subsets of  $\mathbb{B}$  by  $\mathbb{P}^{fs}(\mathbb{B})$ . We note that  $\hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B})$  is well-defined for each  $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$  and  $\pi^\mathbb{B} \in \mathbb{P}^{fs}(\mathbb{B})$  because

$$\hat{c}^-(\pi^\mathbb{A}, \pi^\mathbb{B}) = \alpha^{(1)} \int_{\mathbb{A}} c^-(a, b^{(1)}) \pi^\mathbb{A}(da) + \dots + \alpha^{(K)} \int_{\mathbb{A}} c^-(a, b^{(K)}) \pi^\mathbb{A}(da) > -\infty,$$

see condition (iv) from Definition 3.1, where  $K = 1, 2, \dots$ ,  $\{\alpha^{(1)}, \dots, \alpha^{(K)}\} \subset \mathbb{R}_+$  and  $\{b^{(1)}, \dots, b^{(K)}\} \subset B(x)$  satisfy  $\alpha^{(1)} + \dots + \alpha^{(K)} = 1$  and  $\pi^\mathbb{B}(B) = \alpha^{(1)} \mathbf{I}\{b^{(1)} \in B\} + \dots + \alpha^{(K)} \mathbf{I}\{b^{(K)} \in B\}$  for each  $B \in \mathcal{B}(\mathbb{B})$ . We note that  $\mathbf{I}\{b^{(i)} \in B\} = 1$  when  $b^{(i)} \in B$  and  $\mathbf{I}\{b^{(i)} \in B\} = 0$  otherwise;  $i = 1, \dots, K$ . Remark also that  $\hat{c}^\sharp(\pi^\mathbb{A})$  and  $\hat{c}^\flat(\pi^\mathbb{B})$  are well-defined as in (2.1) for each  $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$  and  $\pi^\mathbb{B} \in \mathbb{P}^{fs}(\mathbb{B})$ . Moreover, the following equality holds:

$$\hat{c}^\flat(\pi^\mathbb{B}) = \inf_{\pi_*^\mathbb{A} \in \mathbb{P}(\mathbb{A})} \hat{c}(\pi_*^\mathbb{A}, \pi^\mathbb{B}), \quad (3.1)$$

for each  $\pi^\mathbb{B} \in \mathbb{P}^{fs}(\mathbb{B})$ ; cf. Theorem 2.9.

The lopsided minimax theorem has the following formulation.

**Theorem 3.3.** *Let a lopsided two-person zero-sum game  $\{\mathbb{A}, \mathbb{B}, c\}$  introduced in Definition 3.1 satisfy the following assumptions:*

- (i) *for each  $b \in \mathbb{B}$  the function  $a \rightarrow c(a, b)$  is lower semi-continuous;*
- (ii) *there exists  $b_0 \in \mathbb{B}$  such that the function  $a \rightarrow c(a, b_0)$  is inf-compact on  $\mathbb{A}$ .*

*Then the following equality hold:*

$$\sup_{\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}}) = \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}^{\sharp}(\pi^{\mathbb{A}}) (= V). \quad (3.2)$$

*Moreover, the set  $\mathbb{P}_V^{\sharp}(\mathbb{A})$  is a nonempty convex compact subset of  $\mathbb{P}(\mathbb{A})$  and, therefore, there exists  $\pi_*^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  such that  $\hat{c}^{\sharp}(\pi_*^{\mathbb{A}}) = \min_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}^{\sharp}(\pi^{\mathbb{A}}) = V$ . If additionally  $\mathbb{B}$  is a measurable space and  $c$  is a measurable function on  $\mathbb{A} \times \mathbb{B}$ , then*

$$\sup_{\pi^{\mathbb{B}} \in \Delta(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}}) = \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}^{\sharp}(\pi^{\mathbb{A}}) \quad (3.3)$$

*for each  $\Delta(\mathbb{B}) \subset \mathbb{P}(\mathbb{B})$  such that a)  $\mathbb{P}^{fs}(\mathbb{B}) \subset \Delta(\mathbb{B})$  and b)  $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$  is well-defined for each  $\pi^{\mathbb{B}} \in \Delta(\mathbb{B})$  and  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ .*

**Remark 3.4.** Both decision sets may be neither compact nor convex in Theorem 3.3. This theorem is generalized version of Proposition I.2.2 from Mertens et al. [17, p. 18] and Aubin and Ekeland [2, Theorem 6.2.7] for Borel subsets of Polish spaces (see also minimax theorems from Fan [4], Kneser [15], Mertens [16], and Sion [20]); see Appendix A. Note that equality (3.3) is proved in Feinberg et al. [11, Theorem 5.11].

**Remark 3.5.** Let assumptions of Theorem 3.3 hold. If, in addition,

$$\sup_{\pi^{\mathbb{B}} \in \mathbb{P}^U(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}}) \leq \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}}), \quad (3.4)$$

then equality (3.3) imply the existence of the weak value  $v_w$  of the game  $\{\mathbb{A}, \mathbb{B}, c\}$ . Moreover,

$$\sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}}) = \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}^{\sharp}(\pi^{\mathbb{A}}). \quad (3.5)$$

In particular, (3.4) and (3.5) hold if  $\hat{c}^b(\pi^{\mathbb{B}}) = -\infty$  for all  $\pi^{\mathbb{B}} \in \mathbb{P}^U$ .

Let  $\mathbb{F}(S)$  denote be the family of all finite subsets of a set  $S$ . The proof of Theorem 3.3 uses the following lemma.

**Lemma 3.6.** (Aubin and Ekeland [2, Theorem 6.2.2]) *Let  $A$  and  $B$  be nonempty convex subsets of vector spaces and  $f : A \times B \rightarrow \mathbb{R}$  be a function such that  $a \rightarrow f(a, b)$  is convex and  $b \rightarrow f(a, b)$  is concave. Then*

$$\sup_{b \in B} \inf_{a \in A} f(a, b) = \sup_{F \in \mathbb{F}(B)} \inf_{a \in A} \max_{b \in F} f(a, b). \quad (3.6)$$

*Proof of Theorem 3.3.* Let a lopsided two-person zero-sum game  $\{\mathbb{A}, \mathbb{B}, c\}$  introduced in Definition 3.1 satisfy assumptions (i,ii) of Theorem 3.3. According to Feinberg et al. [11, p. 18], the following statements hold:

- (i<sub>1</sub>) the sets  $\mathbb{P}_{<+\infty}^\sharp(\mathbb{A})$  and  $\mathbb{P}^{fs}(\mathbb{B})$  are nonempty and convex;
- (i<sub>2</sub>) the function  $\hat{c} : \mathbb{P}_{<+\infty}^\sharp(\mathbb{A}) \times \mathbb{P}^{fs}(\mathbb{B}) \rightarrow \mathbb{R}$  is well-defined and affine in each variable;
- (i<sub>3</sub>) the function  $\hat{c}(\cdot, \pi^\mathbb{B}) : \mathbb{P}(\mathbb{A}) \rightarrow \mathbb{R} \cup \{+\infty\}$  is lower semi-continuous for each  $\pi^\mathbb{B} \in \mathbb{P}^{fs}(\mathbb{B})$ ;
- (i<sub>4</sub>) the function  $\hat{c}(\cdot, b_0) : \mathbb{P}(\mathbb{A}) \rightarrow \mathbb{R} \cup \{+\infty\}$  is inf-compact on  $\mathbb{P}(\mathbb{A})$ .

We observe that in Feinberg et al. [11, Theorem 5.11, pp. 17–20] it was assumed that  $\mathbb{B}$  is a Borel subset of a Polish space and  $c : \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{R}$  is a measurable function, but these properties were not used.

Let us prove that equality (3.2) holds. In view of inequality (2.4) (cf. Remark 2.10),

$$\sup_{\pi^\mathbb{B} \in \mathbb{P}^{fs}(\mathbb{B})} \hat{c}^\flat(\pi^\mathbb{B}) \leq \inf_{\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^\mathbb{A})$$

Therefore, it is sufficient to prove that

$$\inf_{\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^\mathbb{A}) \leq \sup_{\pi^\mathbb{B} \in \mathbb{P}^{fs}(\mathbb{B})} \hat{c}^\flat(\pi^\mathbb{B}). \quad (3.7)$$

Let us prove inequality (3.7). We denote the left-hand side of inequality (3.7) by  $V^\sharp$  and the right-hand side of inequality (3.7) by  $V^\flat$ . For each  $\pi^\mathbb{B} \in \mathbb{P}^{fs}(\mathbb{B})$  the following equality holds:

$$\hat{c}^\flat(\pi^\mathbb{B}) = \inf_{a \in \mathbb{A}} \hat{c}(a, \pi^\mathbb{B}) = \inf_{\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})} \hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B}), \quad (3.8)$$

see (3.1). Remark 2.7 and the definition of the set  $\mathbb{P}_{<+\infty}^\sharp(\mathbb{A})$  imply that each pure strategy of Player I belongs to  $\mathbb{P}_{<+\infty}^\sharp(\mathbb{A}) \subset \mathbb{P}(\mathbb{A})$ . Therefore, (3.8) implies

$$\hat{c}^\flat(\pi^\mathbb{B}) = \inf_{\pi^\mathbb{A} \in \mathbb{P}_{<+\infty}^\sharp(\mathbb{A})} \hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B}),$$

for each  $\pi^\mathbb{B} \in \mathbb{P}^{fs}(\mathbb{B})$ . Thus,

$$V^\flat = \sup_{\pi^\mathbb{B} \in \mathbb{P}^{fs}(\mathbb{B})} \inf_{\pi^\mathbb{A} \in \mathbb{P}_{<+\infty}^\sharp(\mathbb{A})} \hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B}). \quad (3.9)$$

Lemma 3.6, being applied to  $A = \mathbb{P}_{<+\infty}^\sharp(\mathbb{A})$ ,  $B = \mathbb{P}^{fs}(\mathbb{B})$ , and  $f = \hat{c}$ , and properties (i<sub>1</sub>) and (i<sub>2</sub>) yield

$$\sup_{\pi^\mathbb{B} \in \mathbb{P}^{fs}(\mathbb{B})} \inf_{\pi^\mathbb{A} \in \mathbb{P}_{<+\infty}^\sharp(\mathbb{A})} \hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B}) = \sup_{F \in \mathbb{F}(\mathbb{P}^{fs}(\mathbb{B}))} \inf_{\pi^\mathbb{A} \in \mathbb{P}_{<+\infty}^\sharp(\mathbb{A})} \max_{\pi^\mathbb{B} \in F} \hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B}). \quad (3.10)$$

Let  $\mathbb{F}_0(\mathbb{P}^{fs}(\mathbb{B}))$  denote the family of all finite subsets of  $\mathbb{P}^{fs}(\mathbb{B})$  containing the pure strategy of Player II concentrated at the point  $b_0 \in \mathbb{B}$ , whose existence is stated in assumption (ii). Since  $\mathbb{P}_{<+\infty}^\sharp(\mathbb{A}) \subset \mathbb{P}(\mathbb{A})$  and  $\mathbb{F}_0(\mathbb{P}^{fs}(\mathbb{B})) \subset \mathbb{F}(\mathbb{P}^{fs}(\mathbb{B}))$ , then

$$V^* := \sup_{F \in \mathbb{F}_0(\mathbb{P}^{fs}(\mathbb{B}))} \inf_{\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})} \max_{\pi^\mathbb{B} \in F} \hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B}) \leq \sup_{F \in \mathbb{F}(\mathbb{P}^{fs}(\mathbb{B}))} \inf_{\pi^\mathbb{A} \in \mathbb{P}_{<+\infty}^\sharp(\mathbb{A})} \max_{\pi^\mathbb{B} \in F} \hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B}). \quad (3.11)$$

If

$$V^\sharp \leq V^*, \quad (3.12)$$

then formulae (3.9)–(3.11) imply (3.7). Therefore, if inequality (3.12) holds, then equality (3.2) holds.

Let us prove (3.12). Statements  $(i_3, i_4)$  yield that the function  $\max_{\pi^{\mathbb{B}} \in F} \hat{c}(\cdot, \pi^{\mathbb{B}})$  is inf-compact on  $\mathbb{P}(\mathbb{A})$  for each  $F \in \mathbb{F}_0(\mathbb{P}^{fs}(\mathbb{B}))$ . Therefore, for each  $F \in \mathbb{F}_0(\mathbb{P}^{fs}(\mathbb{B}))$  there exists  $\pi_F^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  such that  $\pi_F^{\mathbb{A}} \in \arg \min_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \{\max_{\pi^{\mathbb{B}} \in F} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})\}$ . The definition of  $V^*$  given in (3.11) implies that  $\pi_F^{\mathbb{A}} \in \cap_{\pi^{\mathbb{B}} \in F} \mathcal{D}_{\hat{c}(\cdot, \pi^{\mathbb{B}})}(V^*)$  for each  $F \in \mathbb{F}_0(\mathbb{P}^{fs}(\mathbb{B}))$ . Thus, for each  $F \in \mathbb{F}_0(\mathbb{P}^{fs}(\mathbb{B}))$ ,

$$\cap_{\pi^{\mathbb{B}} \in F} \mathcal{D}_{\hat{c}(\cdot, \pi^{\mathbb{B}})}(V^*) \neq \emptyset. \quad (3.13)$$

Statement  $(i_3)$  and Remark 2.2 yield that the set  $\mathcal{D}_{\hat{c}(\cdot, \pi^{\mathbb{B}})}(V^*)$  is closed for each  $\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(\mathbb{B})$ . Statement  $(i_4)$  implies that the set  $\mathcal{D}_{\hat{c}(\cdot, b_0)}(V^*)$  is compact. Since, as follows from (3.13), the collection  $\{\mathcal{D}_{\hat{c}(\cdot, \pi^{\mathbb{B}})}(V^*) \cap \mathcal{D}_{\hat{c}(\cdot, b_0)}(V^*)\}_{\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(\mathbb{B})}$  of closed subsets of the compact set  $\mathcal{D}_{\hat{c}(\cdot, b_0)}(V^*)$  satisfies the finite intersection property, then this collection has a nonempty intersection, that is, there exists  $\pi_*^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  such that  $\pi_*^{\mathbb{A}} \in \cap_{\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(\mathbb{B})} \mathcal{D}_{\hat{c}(\cdot, \pi^{\mathbb{B}})}(V^*)$ ; see for example Reed and Simon [19, p. 98]. Thus  $\hat{c}(\pi_*^{\mathbb{A}}, \pi^{\mathbb{B}}) \leq V^*$  for all  $\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(\mathbb{B})$ , and therefore

$$\sup_{\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(\mathbb{B})} \hat{c}(\pi_*^{\mathbb{A}}, \pi^{\mathbb{B}}) \leq V^*. \quad (3.14)$$

We note that the inequality

$$\hat{c}^\sharp(\pi_*^{\mathbb{A}}) \leq \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(\mathbb{B})} \hat{c}(\pi_*^{\mathbb{A}}, \pi^{\mathbb{B}}) \quad (3.15)$$

holds because each pure strategy of Player II belongs to  $\mathbb{P}^{fs}(\mathbb{B})$ .

Inequalities (3.14) and (3.15) and the definition of  $V^\sharp$  imply inequality (3.12), which yields inequality (3.7). Thus, equality (3.2) holds.

Let us prove that the set  $\mathbb{P}_V^\sharp(\mathbb{A})$  is a nonempty convex compact subset of  $\mathbb{P}(\mathbb{A})$ . The nonemptiness of the set  $\mathbb{P}_V^\sharp(\mathbb{A})$  follows from (3.14) and (3.15) because  $V^* = V^\sharp = V$ , where  $V$  is defined in (3.2). Moreover, according to properties  $(i_2-i_4)$ , the set  $\mathcal{D}_{\hat{c}(\cdot, b_0)}(V)$  is a convex compact subset of  $\mathbb{P}(\mathbb{A})$  and the set  $\mathcal{D}_{\hat{c}(\cdot, \pi^{\mathbb{B}})}(V)$  is a convex closed subset of  $\mathbb{P}(\mathbb{A})$  for each  $\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(\mathbb{B})$ . Therefore,

$$\mathbb{P}_V^\sharp(\mathbb{A}) = \cap_{\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(\mathbb{B})} \mathcal{D}_{\hat{c}(\cdot, \pi^{\mathbb{B}})}(V)$$

is a nonempty convex compact subset of  $\mathbb{P}(\mathbb{A})$ . In particular, since  $\mathbb{P}_V^\sharp(\mathbb{A})$  is a nonempty set, then there exists  $\pi_*^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  such that  $\hat{c}^\sharp(\pi_*^{\mathbb{A}}) = \min_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^{\mathbb{A}}) = V$ .

Let additionally  $\mathbb{B}$  is a measurable space and  $c$  is a measurable function on  $\mathbb{A} \times \mathbb{B}$ . Let us prove that equality (3.3) holds; cf. Feinberg et al. [11, Theorem 5.11]. Since  $\mathbb{P}^{fs}(\mathbb{B}) \subset \Delta(\mathbb{B})$  and  $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$  is well-defined for each  $\pi^{\mathbb{B}} \in \Delta(\mathbb{B})$  and  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ , then

$$\sup_{\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}}) \leq \sup_{\pi^{\mathbb{B}} \in \Delta(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}}) \leq \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^{\mathbb{A}}). \quad (3.16)$$

Therefore, equality (3.2) and inequalities (3.16) imply equality (3.3).  $\square$

## 4 Uncertainty Neutralization

In this section we introduce some auxiliary results and describe the assumptions for a two-person zero-sum game with possibly noncompact action sets and unbounded payoffs that neutralize the uncertainty. Let  $\{\mathbb{A}, \mathbb{B}, c\}$  be a two-person zero-sum game introduced in Definition 2.3.

**Remark 4.1.** For each  $a \in \mathbb{A}$  we write  $c^\sharp(a)$  instead of  $\hat{c}^\sharp(\pi^\mathbb{A})$ , when  $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$  is a pure strategy for Player I concentrated at the point  $a$ . Similarly, for each  $b \in \mathbb{B}$  we write  $c^\flat(b)$  instead of  $\hat{c}^\flat(\pi^\mathbb{B})$ , when  $\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})$  is a pure strategy for Player II concentrated at the point  $b$ . Note that the following equalities hold:

$$c^\flat(b) = \inf_{a^* \in \mathbb{A}} c(a^*, b) \quad \text{and} \quad c^\sharp(a) = \sup_{b^* \in \mathbb{B}} c(a, b^*), \quad (4.1)$$

for each  $a \in \mathbb{A}$  and  $b \in \mathbb{B}$ . Moreover, the following inequalities hold:

$$-\infty < c^\flat(b) \leq c(a, b) \leq c^\sharp(a) < +\infty, \quad (4.2)$$

for each  $a \in \mathbb{A}$  and  $b \in \mathbb{B}$ , where the first and the last inequalities follow from conditions (iv) and (v) from Definition 2.3 of the game  $\{\mathbb{A}, \mathbb{B}, c\}$ .

We shall provide sufficient conditions for the following statement:

$$\hat{c}^\flat(\pi^\mathbb{B}) = -\infty, \text{ if } \pi^\mathbb{B} \in \mathbb{P}^U(\mathbb{B}). \quad (4.3)$$

**Remark 4.2.** Statement (4.3) holds if and only if  $\mathbb{P}_\beta^\flat(\mathbb{B}) \subset \mathbb{P}^S(\mathbb{B})$  for each  $\beta \in \mathbb{R}$ , because “ $A \Rightarrow B$ ”  $\Leftrightarrow$  “ $\neg B \Rightarrow \neg A$ ” with  $A := “\pi^\mathbb{B} \in \mathbb{P}^U(\mathbb{B})”$  and  $B := “\hat{c}^\flat(\pi^\mathbb{B}) = -\infty”$  for a fixed  $\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})$ . Here we note that  $\neg B = “\pi^\mathbb{B} \in \mathbb{P}_\beta^\flat(\mathbb{B})$  for some  $\beta \in \mathbb{R}”$  and  $\neg A = “\pi^\mathbb{B} \in \mathbb{P}^S(\mathbb{B})”$ .

If the function  $c : \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{R}$  is bounded from above, then the standard definition of the lower bound is

$$v^\flat = \sup_{\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})} \inf_{\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})} \hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B}) = \sup_{\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})} \hat{c}^\flat(\pi^\mathbb{B}).$$

For the general two-person zero-sum game  $\{\mathbb{A}, \mathbb{B}, c\}$  introduced in Definition 2.3, if (4.3) holds then, according to (2.5) and (2.7),

$$v^\flat = \sup_{\pi^\mathbb{B} \in \mathbb{P}^S(\mathbb{B})} \hat{c}^\flat(\pi^\mathbb{B}) = \max\{-\infty; \sup_{\pi^\mathbb{B} \in \mathbb{P}^S(\mathbb{B})} \hat{c}^\flat(\pi^\mathbb{B})\} = \sup_{\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})} \hat{c}^\flat(\pi^\mathbb{B}).$$

Therefore  $v^\flat = \sup_{\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})} \hat{c}^\flat(\pi^\mathbb{B})$  in the both cases, and the lower value can be defined in the same supinf expression as a one of the equivalent definitions of the value where the function  $c$  is bounded. This explains the term uncertainty neutralization. The similar remark is applicable to the upper value  $v^\sharp$ .

Consider the following assumptions:

**(MbW)** there exist  $\gamma_\mathbb{B} \in (0, 1)$ ,  $L_\mathbb{B} > 0$ , and  $\pi_0^\mathbb{A} \in \mathbb{P}(\mathbb{A})$  such that for each  $b \in \mathbb{B}$

$$\int_{\mathbb{A}} \max\{0, c^\sharp(a)\} \pi_0^\mathbb{A}(da) < +\infty \quad \text{and} \quad \hat{c}^-(\pi_0^\mathbb{A}, b) \leq \gamma_\mathbb{B} c^\flat(b) + L_\mathbb{B}; \quad (4.4)$$

**(Mb)** there exist  $\gamma_\mathbb{B} \in (0, 1)$ ,  $L_\mathbb{B} > 0$ , and  $a_0 \in \mathbb{A}$  such that for each  $b \in \mathbb{B}$

$$c^-(a_0, b) \leq \gamma_\mathbb{B} c^\flat(b) + L_\mathbb{B}. \quad (4.5)$$

**Remark 4.3.** Assumption (Mb) implies Assumption (MbW), because each pure strategy  $a_0 \in \mathbb{A}$  for Player I is identified with the mixed one satisfying  $c^\sharp(a_0) < +\infty$ ; see Definition 2.3(v).

**Remark 4.4.** Let  $\{\mathbb{A}, \mathbb{B}, c\}$  be a two-person zero-sum game introduced in Definition 2.3. If  $c$  is bounded from below on  $\mathbb{A} \times \mathbb{B}$ , then Assumption (Mb) holds and, therefore, Assumption (MbW) holds, because the function  $c^\flat$  is bounded from below on  $\mathbb{B}$  and the function  $c^-$  takes non-positive values; see also Remark 4.3. Moreover,  $\mathbb{P}^S(\mathbb{A}) = \mathbb{P}(\mathbb{A})$  and  $\mathbb{P}^S(\mathbb{B}) = \mathbb{P}(\mathbb{B})$ .

**Remark 4.5.** Let  $\{\mathbb{A}, \mathbb{B}, c\}$  be a two-person zero-sum game introduced in Definition 2.3. If the set  $\mathbb{B}$  is compact and the function  $c$  is lower semi-continuous, then Assumption (Mb) holds and, therefore, Assumption (MbW) holds; see Remark 4.4. Indeed, Berge's theorem yields that the function  $c^\flat : \mathbb{B} \rightarrow \mathbb{R}$  is lower semi-continuous (Berge [3, p. 116], Feinberg et al. [7, p. 255], Hu and Papageorgiou [13, p. 83]) and, therefore, it is bounded from below on a compact set  $\mathbb{B}$ , as follows from the Weierstrass extreme value theorem for lower semi-continuous functions. Then, according to Remark 4.4, Assumption (Mb) holds and, therefore, Assumption (MbW) holds; see also Remark 4.3. Moreover,  $\mathbb{P}^S(\mathbb{A}) = \mathbb{P}(\mathbb{A})$  and  $\mathbb{P}^S(\mathbb{B}) = \mathbb{P}(\mathbb{B})$ .

In the following example we describe the two-person zero-sum game  $\{\mathbb{A}, \mathbb{B}, c\}$  with noncompact action sets and unbounded payoffs satisfying Assumption (Mb) and, therefore, which satisfies Assumption (MbW); see Remark 4.3.

**Example 4.6.** Let  $\mathbb{A} = \mathbb{B} = \mathbb{R}$ ,  $c(a, b) = a^2 - b^2$ ,  $(a, b) \in \mathbb{R}^2$ . Then the game  $\{\mathbb{A}, \mathbb{B}, c\}$  satisfies Assumption (Mb) and, therefore, it satisfies Assumption (MbW); see Remark 4.3. Indeed, if we consider arbitrary  $a_0 \in \mathbb{R}$ ,  $\gamma_{\mathbb{B}} \in (0, 1)$  and set  $L_{\mathbb{B}} := a_0^2$ , then

$$c^-(a_0, b) \leq c(a_0, b) = a_0^2 - b^2 \leq -\gamma_{\mathbb{B}} b^2 + a_0^2 = \gamma_{\mathbb{B}} c^\flat(b) + L_{\mathbb{B}},$$

for each  $b \in \mathbb{R}$ , because  $c^\flat(b) = -b^2$  for each  $b \in \mathbb{R}$ .

Example 4.6 admits the following interpretation in the form of a simple game of timing (see Yanovskaya [21, Section 6]) with noncompact decision sets. Two teams work on a project consisting of two independent tasks, each performed by one of the teams. The project should be completed on a target date. The project is completed when both tasks are completed, and they should be completed simultaneously. The penalty, in the amount of  $t^2$  paid to another team for completing its task by  $t$  units of time later or earlier than the target date, creates incentives to the teams to complete their tasks exactly on time. Of course, there are other payoff functions including  $|t|$  that provide incentives to achieve the same goal.

The following lemma and its corollary describe sufficient conditions for (4.3); see also Remark 4.2.

**Lemma 4.7.** Let a two-person zero-sum game  $\{\mathbb{A}, \mathbb{B}, c\}$  introduced in Definition 2.3 satisfy Assumption (MbW). Then

$$\hat{c}^\flat(\pi^{\mathbb{B}}) \leq \gamma_{\mathbb{B}} \int_{\mathbb{A}} \int_{\mathbb{B}} c^-(a, b) \pi^{\mathbb{B}}(db) \pi^{\mathbb{A}}(da) + L_{\mathbb{B}} + \int_{\mathbb{A}} \max\{0, c^\sharp(a)\} \pi_0^{\mathbb{A}}(da), \quad (4.6)$$

for each  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  and  $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$ . Furthermore, if  $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$  satisfies  $\hat{c}^\flat(\pi^{\mathbb{B}}) > -\infty$ , then  $\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})$ .

**Remark 4.8.** Example 5.5 below demonstrates that under the condition, that the function  $(b, a) \rightarrow c(a, b)$  is  $\mathbb{K}$ -inf-compact on  $\mathbb{B} \times \mathbb{A}$  (see Feinberg et al. [9, Definition 1.1]), which is a stronger condition than the assumptions (i) and (ii) of Theorem 5.1 below, it is possible that  $\hat{c}^\flat(\pi^{\mathbb{B}}) > -\infty$  for some  $\pi^{\mathbb{B}} \in \mathbb{P}^U(\mathbb{B})$ , when Assumption (MbW) does not hold.



*Proof of Lemma 4.7.* Let us fix arbitrary  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ ,  $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$  and prove that inequality (4.6) holds. Note that

$$\inf_{a \in \mathbb{A}} c^-(a, b) = \min\{c^b(b), 0\} \quad (4.7)$$

for each  $b \in \mathbb{B}$ . Indeed, if  $c^b(b) \geq 0$ , then  $c(a, b) \geq 0$  for each  $a \in \mathbb{A}$ . Otherwise, if  $c^b(b) < 0$ , then the set  $A^-(b) := \{a \in \mathbb{A} : c(a, b) < 0\} = \{a \in \mathbb{A} : c^-(a, b) < 0\}$  is nonempty and

$$\inf_{a \in \mathbb{A}} c^-(a, b) = \inf_{a \in A^-(b)} c^-(a, b) = \inf_{a \in A^-(b)} c(a, b) = \inf_{a \in \mathbb{A}} c(a, b) = c^b(b),$$

which follow from the basic properties of infima and the definition of  $A^-(b)$ .

Since  $\inf_{a^* \in \mathbb{A}} c^-(a^*, b) \leq c^-(a, b)$  for each  $a \in \mathbb{A}$  and  $b \in \mathbb{B}$ , then equality (4.7) yields

$$\int_{\mathbb{B}} \min\{c^b(b), 0\} \pi^{\mathbb{B}}(db) \leq \int_{\mathbb{A}} \int_{\mathbb{B}} c^-(a, b) \pi^{\mathbb{B}}(db) \pi^{\mathbb{A}}(da). \quad (4.8)$$

The second inequality in (4.4) implies that

$$-L_{\mathbb{B}} + \hat{c}^-(\pi_0^{\mathbb{A}}, b) \leq \gamma_{\mathbb{B}} \min\{c^b(b), 0\}$$

for each  $b \in \mathbb{B}$  because  $-L_{\mathbb{B}} < 0$  and  $c^-(a, b) \leq 0$  for each  $a \in \mathbb{A}$  and  $b \in \mathbb{B}$ . Thus,

$$-L_{\mathbb{B}} + \hat{c}^-(\pi_0^{\mathbb{A}}, \pi^{\mathbb{B}}) \leq \gamma_{\mathbb{B}} \int_{\mathbb{B}} \min\{c^b(b), 0\} \pi^{\mathbb{B}}(db), \quad (4.9)$$

and

$$\hat{c}^+(\pi_0^{\mathbb{A}}, \pi^{\mathbb{B}}) \leq \int_{\mathbb{A}} \max\{0, c^{\sharp}(a)\} \pi_0^{\mathbb{A}}(da) < +\infty, \quad (4.10)$$

where the first inequality follows from (4.2) and the second inequality follows from (4.4).

Inequalities (4.9) and (4.10) yield that

$$\hat{c}(\pi_0^{\mathbb{A}}, \pi^{\mathbb{B}}) \leq \gamma_{\mathbb{B}} \int_{\mathbb{B}} \min\{c^b(b), 0\} \pi^{\mathbb{B}}(db) + L_{\mathbb{B}} + \int_{\mathbb{A}} \max\{0, c^{\sharp}(a)\} \pi_0^{\mathbb{A}}(da). \quad (4.11)$$

Since  $\hat{c}^b(\pi^{\mathbb{B}}) \leq \hat{c}(\pi_0^{\mathbb{A}}, \pi^{\mathbb{B}}) < +\infty$ , then inequalities (4.8) and (4.11) imply inequality (4.6).

Now let  $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$  satisfy the inequality  $\hat{c}^b(\pi^{\mathbb{B}}) > -\infty$ . Observe that  $\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})$ . Indeed, since  $\hat{c}^b(\pi^{\mathbb{B}}) > -\infty$ , then the first inequality in (4.4) and inequality (4.6) imply that

$$-\infty < \int_{\mathbb{A}} \int_{\mathbb{B}} c^-(a, b) \pi^{\mathbb{B}}(db) \pi^{\mathbb{A}}(da),$$

for each  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ , that is,  $\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})$ . □

**Corollary 4.9.** *Let the two-person zero-sum game  $\{\mathbb{A}, \mathbb{B}, c\}$  introduced in Definition 2.3 satisfy Assumption (Mb). Then*

$$\hat{c}^b(\pi^{\mathbb{B}}) \leq \gamma_{\mathbb{B}} \int_{\mathbb{A}} \int_{\mathbb{B}} c^-(a, b) \pi^{\mathbb{B}}(db) \pi^{\mathbb{A}}(da) + L_{\mathbb{B}} + \max\{0, c^{\sharp}(a_0)\} \quad (4.12)$$

for each  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  and  $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$ . Furthermore, if  $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$  satisfies  $\hat{c}^b(\pi^{\mathbb{B}}) > -\infty$ , then  $\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})$ .

*Proof.* According to Remark 4.3, Assumption (Mb) implies Assumption (MbW). Then Lemma 4.7 yields all the statements of Corollary 4.9.  $\square$

Next, we provide sufficient conditions for the statement

$$\hat{c}^\sharp(\pi^\mathbb{A}) = +\infty, \text{ if } \pi^\mathbb{A} \in \mathbb{P}^U(\mathbb{A}). \quad (4.13)$$

**Remark 4.10.** Statement (4.13) holds if and only if  $\mathbb{P}_\alpha^\sharp(\mathbb{A}) \subset \mathbb{P}^S(\mathbb{A})$  for each  $\alpha \in \mathbb{R}$ , because “ $A \Rightarrow B$ ”  $\Leftrightarrow$  “ $\neg B \Rightarrow \neg A$ ” with  $A := “\pi^\mathbb{A} \in \mathbb{P}^U(\mathbb{A})”$  and  $B := “\hat{c}^\sharp(\pi^\mathbb{A}) = +\infty”$  for a fixed  $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$ . Here we note that  $\neg B = “\pi^\mathbb{A} \in \mathbb{P}_\alpha^\sharp(\mathbb{A})$  for some  $\alpha \in \mathbb{R}”$  and  $\neg A = “\pi^\mathbb{A} \in \mathbb{P}^S(\mathbb{A})”$ .

Consider the following assumptions.

**(MaW)** there exist  $\gamma_\mathbb{A} \in (0, 1)$ ,  $L_\mathbb{A} > 0$ , and  $\pi_0^\mathbb{B} \in \mathbb{P}^S(\mathbb{B})$  such that for each  $a \in \mathbb{A}$

$$-\infty < \int_{\mathbb{B}} \min\{0, c^\flat(b)\} \pi_0^\mathbb{B}(db) \quad \text{and} \quad -L_\mathbb{A} + \gamma_\mathbb{A} \hat{c}^\sharp(a) \leq \hat{c}^+(a, \pi_0^\mathbb{B}). \quad (4.14)$$

**(Ma)** there exist  $\gamma_\mathbb{A} \in (0, 1)$ ,  $L_\mathbb{A} > 0$ , and  $b_0 \in \mathbb{B}$  such that for each  $a \in \mathbb{A}$

$$-L_\mathbb{A} + \gamma_\mathbb{A} \hat{c}^\sharp(a) \leq c^+(a, b_0). \quad (4.15)$$

**Remark 4.11.** Assumption (Ma) implies Assumption (MaW), because each pure strategy for Player II is identified with the mixed ones and  $c^\flat(b_0) > -\infty$ ; see Definition 2.3(iv).

**Remark 4.12.** A two-person zero-sum game  $\{\mathbb{A}, \mathbb{B}, c\}$  introduced in Definition 2.3 satisfy Assumption (MaW) (Assumptions (MbW), (Ma), (Mb) respectively) if and only if the game  $\{\mathbb{B}, \mathbb{A}, -c^{\mathbb{A} \leftrightarrow \mathbb{B}}\}$  introduced in Remark 2.4 satisfy Assumption (MbW) (Assumptions (MaW), (Mb), (Ma) respectively).

**Remark 4.13.** Let  $\{\mathbb{A}, \mathbb{B}, c\}$  be a two-person zero-sum game introduced in Definition 2.3. If  $c$  is bounded from above on  $\mathbb{A} \times \mathbb{B}$ , then Assumption (Ma) holds and, therefore, Assumption (MaW) holds because the function  $c^\sharp$  is bounded from above on  $\mathbb{A}$  and the function  $c^+$  takes non-negative values; see Remarks 4.4 and 4.12. Moreover,  $\mathbb{P}^S(\mathbb{A}) = \mathbb{P}(\mathbb{A})$  and  $\mathbb{P}^S(\mathbb{B}) = \mathbb{P}(\mathbb{B})$ .

**Remark 4.14.** Let  $\{\mathbb{A}, \mathbb{B}, c\}$  be a two-person zero-sum game introduced in Definition 2.3. If the set  $\mathbb{A}$  is compact and the function  $c$  is upper semi-continuous, then Assumption (Ma) holds and, therefore, Assumption (MaW) holds. Furthermore,  $\mathbb{P}^S(\mathbb{A}) = \mathbb{P}(\mathbb{A})$  and  $\mathbb{P}^S(\mathbb{B}) = \mathbb{P}(\mathbb{B})$ ; see Remarks 4.13, 2.4, and 4.5.

**Remark 4.15.** Example 4.6 describes the two-person zero-sum game  $\{\mathbb{A}, \mathbb{B}, c\}$  with noncompact action sets and unbounded payoffs satisfying Assumption (Mb). Therefore, this game satisfies Assumption (MbW); see Remark 4.3. Since  $\mathbb{A} = \mathbb{B}$  and  $c(a, b) = -c(b, a)$  for each  $a, b = 1, 2, \dots$  in Example 4.6, then Assumption (Ma) holds and, therefore, Assumption (MaW) holds; see also Remark 4.11.

The following lemma and its corollary describe sufficient conditions for (4.13); see also Remark 4.10.

**Lemma 4.16.** *Let the two-person zero-sum game  $\{\mathbb{A}, \mathbb{B}, c\}$  introduced in Definition 2.3 satisfy Assumption (MaW). Then*

$$\gamma_\mathbb{A} \int_{\mathbb{A}} \int_{\mathbb{B}} c^+(a, b) \pi^\mathbb{B}(db) \pi^\mathbb{A}(da) \leq \hat{c}^\sharp(\pi^\mathbb{A}) + L_\mathbb{A} - \int_{\mathbb{B}} \min\{0, c^\flat(b)\} \pi_0^\mathbb{B}(db) \quad (4.16)$$

*for each  $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$  and  $\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})$ . Furthermore, if  $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$  satisfies  $\hat{c}^\sharp(\pi^\mathbb{A}) < +\infty$ , then  $\pi^\mathbb{A} \in \mathbb{P}^S(\mathbb{A})$ .*

*Proof.* All statements follow from Lemma 4.7, being applied to the game  $\{\mathbb{B}, \mathbb{A}, -c^{\mathbb{A} \leftrightarrow \mathbb{B}}\}$  introduced in Remark 2.4.  $\square$

**Corollary 4.17.** *Let two-person zero-sum game  $\{\mathbb{A}, \mathbb{B}, c\}$  introduced in Definition 2.3 satisfy Assumption (Ma). Then*

$$\gamma_{\mathbb{A}} \int_{\mathbb{A}} \int_{\mathbb{B}} c^+(a, b) \pi^{\mathbb{B}}(db) \pi^{\mathbb{A}}(da) \leq \hat{c}^{\#}(\pi^{\mathbb{A}}) + L_{\mathbb{A}} - \min\{0, c^b(b_0)\} \quad (4.17)$$

*for each  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  and  $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$ . Furthermore, if  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  satisfies  $\hat{c}^{\#}(\pi^{\mathbb{A}}) < +\infty$ , then  $\pi^{\mathbb{A}} \in \mathbb{P}^S(\mathbb{A})$ .*

**Remark 4.18.** Example 5.5 and Remark 5.6 below demonstrate the game  $\{\mathbb{A}, \mathbb{B}, c\}$  satisfying the stronger conditions than assumptions (i–iv) of Corollary 5.3 such that  $\hat{c}^{\#}(\pi^{\mathbb{A}}) < +\infty$  for  $\pi^{\mathbb{A}} \in \mathbb{P}^U(\mathbb{A})$ , when Assumption (MaW) does not hold.

*Proof of Corollary 4.17.* Since Assumption (Ma) implies Assumption (MaW), then Lemma 4.16 yields all the statements of Corollary 4.17.  $\square$

## 5 Existence of a Weak Value

The following Theorem 5.1 and its Corollary 5.3 provide sufficient conditions for the existence of a weak value for a two-person zero-sum game with possibly noncompact action sets and unbounded payoffs and describe the properties of the solution sets under these conditions. For well-defined payoff functions, the proof of the existence of the value is usually based on Sion's theorem (Mertens et al. [17, Theorem I.1.1]), that requires that at least one of the decision sets is compact. In our situation, both decision sets may not be compact. In addition, the payoff function  $c$  may be unbounded above and below, and therefore the payoff function  $\hat{c}$  may be undefined for some pairs of mixed strategies. Because of these reasons, our proof of the existence of the weak value does not use Sion's theorem. In general, a game on the unit square with bounded measurable payoffs may not have a value; see Yanovskaya [21, p. 527] and the references to counterexamples by Ville, by Wald, and by Sion and Wolfe cited there. Therefore, some conditions that neutralize the uncertainty are needed, and Theorem 5.1 requires mild Assumption (MbW) and assumptions (i,ii) of Theorem 3.3.

The following theorem and its corollary are the main results of this section.

**Theorem 5.1.** *Let a two-person zero-sum game  $\{\mathbb{A}, \mathbb{B}, c\}$  introduced in Definition 2.3 satisfy assumptions (i,ii) of Theorem 3.3 and Assumption (MbW). Then the game  $\{\mathbb{A}, \mathbb{B}, c\}$  has the weak value  $v_w$ , that is, the second equality from (2.12) holds. Moreover, the set  $\mathbb{P}_{v_w}^{\#}(\mathbb{A})$  is a nonempty convex compact subset of  $\mathbb{P}(\mathbb{A})$ .*

**Remark 5.2.** Under Theorem 5.1 assumptions, equalities (2.13) hold; see Remark 2.19.

*Proof of Theorem 5.1.* According to Theorem 3.3, it is sufficient to prove the following inequality:

$$\sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}}) \leq \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}}).$$

This inequality follows from Lemma 4.7. Indeed, if on the contrary,  $\sup_{\pi^{\mathbb{B}} \in \mathbb{P}^U(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}}) > \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}})$ , then Lemma 4.7 yields that  $\sup_{\pi^{\mathbb{B}} \in \mathbb{P}^U(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}}) = -\infty$  and, therefore,  $\sup_{\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})} \hat{c}^b(\pi^{\mathbb{B}}) < -\infty$ ; see also (4.3) and Remark 4.2. This is a contradiction.  $\square$

**Corollary 5.3.** *Let a two-person zero-sum game  $\{\mathbb{A}, \mathbb{B}, c\}$  introduced in Definition 2.3 satisfy the following assumptions:*

- (i) *for each  $b \in \mathbb{B}$  the function  $a \rightarrow c(a, b)$  is lower semi-continuous;*
- (ii) *there exists  $b_0 \in \mathbb{B}$  such that the function  $a \rightarrow c(a, b_0)$  is inf-compact on  $\mathbb{A}$ ;*
- (iii) *for each  $a \in \mathbb{A}$  the function  $b \rightarrow c(a, b)$  is upper semi-continuous;*
- (vi) *there exists  $a_0 \in \mathbb{A}$  such that the function  $b \rightarrow c(a_0, b)$  is sup-compact on  $\mathbb{B}$ ;*
- (v) *either Assumption (MaW) or Assumption (MbW) holds.*

*Then the game  $\{\mathbb{A}, \mathbb{B}, c\}$  has the weak value  $v_w$ , that is, the second equality from (2.12) holds. Moreover, the sets  $\mathbb{P}_{v_w}^{\#}(\mathbb{A})$  and  $\mathbb{P}_{v_w}^b(\mathbb{B})$  are nonempty convex compact subsets of  $\mathbb{P}(\mathbb{A})$  and  $\mathbb{P}(\mathbb{B})$  respectively.*

**Remark 5.4.** Example 4.6 describes a two-person zero-sum game with noncompact action sets and unbounded payoffs satisfying the assumptions of Corollary 5.3 and, therefore, assumptions of Theorem 5.1; see Remark 4.12.

*Proof of Corollary 5.3.* The existence of a weak value follows from Theorem 5.1, being applied either to the game  $\{\mathbb{A}, \mathbb{B}, c\}$ , if Assumption (MbW) holds, or to the game  $\{\mathbb{B}, \mathbb{A}, -c^{\mathbb{A} \leftrightarrow \mathbb{B}}\}$  (see Remark 2.4), if Assumption (MaW) holds. The least statements of Corollary 5.3 follow from Theorem 3.3, being applied twice to the games  $\{\mathbb{A}, \mathbb{B}, c\}$  and  $\{\mathbb{B}, \mathbb{A}, -c^{\mathbb{A} \leftrightarrow \mathbb{B}}\}$  respectively.  $\square$

The following example demonstrates that under the condition, that the function  $(b, a) \rightarrow c(a, b)$  is  $\mathbb{K}$ -inf-compact on  $\mathbb{B} \times \mathbb{A}$ , which is a stronger condition than the assumptions of Theorem 3.3, it is possible that  $\hat{c}^b(\pi^{\mathbb{B}}) > -\infty$  for some  $\pi^{\mathbb{B}} \in \mathbb{P}^U(\mathbb{B})$ . Therefore, Assumption (MbW) in Theorem 5.1 is essential.

**Example 5.5.** (Feinberg et al [11, Example 5.14]) The function  $(b, a) \rightarrow c(a, b)$  is  $\mathbb{K}$ -inf-compact on  $\mathbb{B} \times \mathbb{A}$ , the function  $(a, b) \rightarrow c(a, b)$  is  $\mathbb{K}$ -sup-compact on  $\mathbb{A} \times \mathbb{B}$ , and there exists  $\pi^{\mathbb{B}} \in \mathbb{P}^U(\mathbb{B})$  such that  $\hat{c}^b(\pi^{\mathbb{B}}) > -\infty$  (see Definitions 8.1 and 8.2 for the definition of  $\mathbb{K}$ -inf-compact and  $\mathbb{K}$ -sup-compact functions. In particular, if  $\mathbb{A}$  and  $\mathbb{B}$  are countable sets, as takes place in this example, a function  $f : \mathbb{A} \times \mathbb{B} \rightarrow R$  is called  $\mathbb{K}$ -inf-compact ( $\mathbb{K}$ -inf-compact) on  $\mathbb{A} \times \mathbb{B}$  if for each  $a \in \mathbb{A}$  the function  $f(a, \cdot)$  is inf-compact (sup-compact) on  $\mathbb{B}$ ).

Let us set  $\mathbb{A} := \mathbb{B} := \{1, 2, \dots\}$ ,  $c(a, b) := 6^a 4^b \mathbf{I}\{b < a\} - 6^b 4^a \mathbf{I}\{a < b\}$ ,  $\pi^{\mathbb{B}}(\{b\}) := \frac{11}{12^b}$ ,  $b = 1, 2, \dots$ . We consider the discrete metrics on  $\mathbb{A}$  and  $\mathbb{B}$ .

The function  $(b, a) \rightarrow c(a, b)$  is  $\mathbb{K}$ -inf-compact on  $\mathbb{B} \times \mathbb{A}$  because  $c(a, b) \rightarrow +\infty$ , as  $a \rightarrow \infty$ , for each  $b = 1, 2, \dots$ . Here we note that a set  $K \subset \mathbb{B}$  is compact if and only if  $K$  is finite. The function  $(a, b) \rightarrow c(a, b)$  is  $\mathbb{K}$ -sup-compact on  $\mathbb{A} \times \mathbb{B}$  because  $c(a, b) \rightarrow -\infty$ , as  $b \rightarrow \infty$ , for each  $a = 1, 2, \dots$ .

We notice that for each  $b = 1, 2, \dots$

$$\begin{aligned}\hat{c}^-(a, \pi^{\mathbb{B}}) &= - \sum_{b=a+1}^{\infty} 6^b 4^a \frac{11}{12^b} = -11 \cdot 4^a \sum_{b=a+1}^{\infty} \frac{1}{2^b} = -11 \cdot 2^a, \\ \hat{c}^+(a, \pi^{\mathbb{B}}) &= \sum_{b=1}^{a-1} 6^a 4^b \frac{11}{12^b} = 11 \cdot 6^a \sum_{b=1}^{a-1} \frac{1}{3^b} = \frac{11}{2} 6^a - \frac{33}{2} 2^a.\end{aligned}$$

Therefore,  $\hat{c}(a, \pi^{\mathbb{B}}) = \frac{11}{2} 6^a - \frac{55}{2} 2^a$  for each  $a = 1, 2, \dots$ . Since  $\hat{c}(a, \pi^{\mathbb{B}}) \rightarrow +\infty$ , as  $a \rightarrow \infty$ , then  $\hat{c}^b(\pi^{\mathbb{B}}) = \inf_{a \in \mathbb{A}} \hat{c}(a, \pi^{\mathbb{B}}) > -\infty$ .

Let us set  $\pi^{\mathbb{A}}(\{a\}) := \frac{1}{2^a}$ ,  $a = 1, 2, \dots$ . Since  $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  and

$$\begin{aligned}\hat{c}^-(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) &= - \sum_{a=1,2,\dots} 11 \cdot 2^a \frac{1}{2^a} = -\infty, \\ \hat{c}^+(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) &= \sum_{a=1,2,\dots} \left( \frac{11}{2} 6^a - \frac{33}{2} 2^a \right) \frac{1}{2^a} = +\infty,\end{aligned}$$

then  $\pi^{\mathbb{B}} \in \mathbb{P}^U(\mathbb{B})$ .

**Remark 5.6.** Since  $\mathbb{A} = \mathbb{B}$  and  $c(a, b) = -c(b, a)$  for each  $a, b = 1, 2, \dots$  in Example 5.5, then, according to Remark 4.8, the function  $(a, b) \rightarrow c(a, b)$  is  $\mathbb{K}$ -sup-compact on  $\mathbb{A} \times \mathbb{B}$  (see Feinberg et al. [11, Definition 1.4]), that is, the game  $\{\mathbb{A}, \mathbb{B}, c\}$  satisfies the stronger conditions than assumptions (i–iv) of Corollary 5.3. Moreover, it is possible that  $\hat{c}^\sharp(\pi^{\mathbb{A}}) < +\infty$  for  $\pi^{\mathbb{A}} := \pi^{\mathbb{B}} \in \mathbb{P}^U(\mathbb{B}) = \mathbb{P}^U(\mathbb{A})$ , when Assumption (MaW) does not hold. Therefore, assumption (v) in Corollary 5.3 is essential; see also Remark 4.8.

## 6 The Existence of a Solution

In this section we provide the definition of a solution for a two-person zero-sum game with possibly non-compact actions and unbounded payoffs. Theorem 6.4 establishes sufficient conditions for the existence of a solution for such a game in safe strategies.

**Definition 6.1.** A pair of mixed strategies  $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}^S(\mathbb{A}) \times \mathbb{P}^S(\mathbb{B})$  for Players I and II is called a solution (saddle point, equilibria) of the game  $\{\mathbb{A}, \mathbb{B}, c\}$  in safe strategies, if

$$\hat{c}(\pi^{\mathbb{A}}, \pi_*^{\mathbb{B}}) \leq \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \leq \hat{c}(\pi_*^{\mathbb{A}}, \pi^{\mathbb{B}}) \quad (6.1)$$

for each  $\pi_*^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$  and  $\pi_*^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$ .

**Remark 6.2.** If the solution  $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}^S(\mathbb{A}) \times \mathbb{P}^S(\mathbb{B})$  of the game  $\{\mathbb{A}, \mathbb{B}, c\}$  in safe strategies exists, then the number  $v = \hat{c}^b(\pi^{\mathbb{B}}) = v_w^b = v_w^\sharp = v^b = v^\sharp = \hat{c}^\sharp(\pi^{\mathbb{A}})$  uniquely defines the value of this game; see Remarks 2.18 and 2.19. Indeed, since  $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}^S(\mathbb{A}) \times \mathbb{P}^S(\mathbb{B})$ , then

$$\inf_{\pi_*^{\mathbb{A}} \in \mathbb{P}^S(\mathbb{A})} \hat{c}^\sharp(\pi_*^{\mathbb{A}}) = \inf_{\pi_*^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi_*^{\mathbb{A}}) = \hat{c}^\sharp(\pi^{\mathbb{A}}) = \hat{c}^b(\pi^{\mathbb{B}}) = \sup_{\pi_*^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \hat{c}^b(\pi_*^{\mathbb{B}}) = \sup_{\pi_*^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})} \hat{c}^\sharp(\pi_*^{\mathbb{B}});$$

see Definition 6.1. Therefore, since  $\mathbb{P}^S(\mathbb{A}) \subset \mathbb{P}_{\pi_*^{\mathbb{B}}}^S(\mathbb{A}) \subset \mathbb{P}(\mathbb{A})$  and  $\mathbb{P}^S(\mathbb{B}) \subset \mathbb{P}_{\pi_*^{\mathbb{A}}}^S(\mathbb{B}) \subset \mathbb{P}(\mathbb{B})$  for each  $\pi_*^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$  and  $\pi_*^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ , then equalities  $v_w^\sharp = v^\sharp = \hat{c}^\sharp(\pi^{\mathbb{A}}) = \hat{c}^b(\pi^{\mathbb{B}}) = v^b = v_w^b = v$  hold.

**Remark 6.3.** Let assumptions of Corollary 5.3 hold. Then, according to Remark 5.2, Corollary 5.3 and Aubin and Ekeland [2, Proposition 1, Chapter 6], there exists the pair of mixed strategies  $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$  with well-defined  $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$  satisfying inequality (6.1) for each  $\pi_*^{\mathbb{A}} \in \mathbb{P}_{\pi^{\mathbb{B}}}^S(\mathbb{A})$  and  $\pi_*^{\mathbb{B}} \in \mathbb{P}_{\pi^{\mathbb{A}}}^S(\mathbb{B})$ . Moreover, a pair of strategies  $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$  with well-defined  $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$  satisfies inequality (6.1) for each  $\pi_*^{\mathbb{A}} \in \mathbb{P}_{\pi^{\mathbb{B}}}^S(\mathbb{A})$  and  $\pi_*^{\mathbb{B}} \in \mathbb{P}_{\pi^{\mathbb{A}}}^S(\mathbb{B})$  if and only if  $\pi^{\mathbb{A}} \in \mathbb{P}_{v_w}^{\sharp}(\mathbb{A})$  and  $\pi^{\mathbb{B}} \in \mathbb{P}_{v_w}^b(\mathbb{B})$ . Therefore, a pair of safe strategies  $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}^S(\mathbb{A}) \times \mathbb{P}^S(\mathbb{B})$  for Players I and II is the solution of the game  $\{\mathbb{A}, \mathbb{B}, c\}$  in safe strategies if and only if  $\pi^{\mathbb{A}} \in \mathbb{P}_{v_w}^{\sharp}(\mathbb{A})$  and  $\pi^{\mathbb{B}} \in \mathbb{P}_{v_w}^b(\mathbb{B})$ .

The following theorem describes sufficient conditions for the existence of a solution.

**Theorem 6.4.** *Let two-person zero-sum game  $\{\mathbb{A}, \mathbb{B}, c\}$  introduced in Definition 2.3 satisfies conditions (i–iv) of Corollary 5.3. Moreover, let Assumptions (MaW) and (MbW) hold. Then the following statements hold:*

- (i) *the game  $\{\mathbb{A}, \mathbb{B}, c\}$  has a value  $v \in \mathbb{R}$  and a solution  $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}_v^{\sharp}(\mathbb{A}) \times \mathbb{P}_v^b(\mathbb{B})$  in safe strategies;*
- (ii) *the sets  $\mathbb{P}_v^{\sharp}(\mathbb{A})$  and  $\mathbb{P}_v^b(\mathbb{B})$  are nonempty convex compact subsets of  $\mathbb{P}(\mathbb{A})$  and  $\mathbb{P}(\mathbb{B})$  respectively. Moreover,  $\mathbb{P}_v^{\sharp}(\mathbb{A}) \subset \mathbb{P}^S(\mathbb{A})$  and  $\mathbb{P}_v^b(\mathbb{B}) \subset \mathbb{P}^S(\mathbb{B})$ ;*
- (iii) *a pair of strategies  $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}^S(\mathbb{A}) \times \mathbb{P}^S(\mathbb{B})$  is a solution of the game  $\{\mathbb{A}, \mathbb{B}, c\}$  in safe strategies if and only if  $\pi^{\mathbb{A}} \in \mathbb{P}_v^{\sharp}(\mathbb{A})$  and  $\pi^{\mathbb{B}} \in \mathbb{P}_v^b(\mathbb{B})$ .*

*Proof.* Corollary 5.3 yields that the game  $\{\mathbb{A}, \mathbb{B}, c\}$  has the weak value  $v_w \in \mathbb{R}$ , that is, equality (2.12) holds. Moreover, the sets  $\mathbb{P}_{v_w}^{\sharp}(\mathbb{A})$  and  $\mathbb{P}_{v_w}^b(\mathbb{B})$  are nonempty convex compact subsets of  $\mathbb{P}(\mathbb{A})$  and  $\mathbb{P}(\mathbb{B})$  respectively. Lemmas 4.7 and 4.16 imply that  $\mathbb{P}_{v_w}^{\sharp}(\mathbb{A}) \subset \mathbb{P}^S(\mathbb{A})$  and  $\mathbb{P}_{v_w}^b(\mathbb{B}) \subset \mathbb{P}^S(\mathbb{B})$  respectively. Therefore, according to Remark 6.3, the game  $\{\mathbb{A}, \mathbb{B}, c\}$  has a solution  $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}_{v_w}^{\sharp}(\mathbb{A}) \times \mathbb{P}_{v_w}^b(\mathbb{B})$  in safe strategies. In particular, the game  $\{\mathbb{A}, \mathbb{B}, c\}$  has a value  $v = v_w$ ; see Remark 6.2. Statement (i) is proved.

Statement (ii) holds, because  $v_w = v$ ,  $\mathbb{P}_{v_w}^{\sharp}(\mathbb{A}) \subset \mathbb{P}^S(\mathbb{A})$ ,  $\mathbb{P}_{v_w}^b(\mathbb{B}) \subset \mathbb{P}^S(\mathbb{B})$ , and the sets  $\mathbb{P}_{v_w}^{\sharp}(\mathbb{A})$  and  $\mathbb{P}_{v_w}^b(\mathbb{B})$  are nonempty convex compact subsets of  $\mathbb{P}(\mathbb{A})$  and  $\mathbb{P}(\mathbb{B})$  respectively.

Since  $v_w = v$ , then, according to Remarks 6.2 and 6.3, a pair of strategies  $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}^S(\mathbb{A}) \times \mathbb{P}^S(\mathbb{B})$  is a solution of the game  $\{\mathbb{A}, \mathbb{B}, c\}$  in safe strategies if and only if  $\pi^{\mathbb{A}} \in \mathbb{P}_v^{\sharp}(\mathbb{A})$  and  $\pi^{\mathbb{B}} \in \mathbb{P}_v^b(\mathbb{B})$ , that is, statement (iii) holds.  $\square$

## 7 Examples

In this section we consider a two-person zero-sum game  $\{\mathbb{A}, \mathbb{B}, c\}$  defined as follows:  $\mathbb{A} = \mathbb{B} := \mathbb{R}_+ = [0, +\infty)$  and  $c(a, b) := \varphi(a - b)$  for each  $a, b \in \mathbb{R}_+$ , where  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function.

**Lemma 7.1.** *The following statements hold:*

- (i) *if  $\varphi(s) \rightarrow +\infty$  as  $s \rightarrow +\infty$ , then the function  $(b, a) \rightarrow c(a, b)$  is  $\mathbb{K}$ -inf-compact on  $\mathbb{B} \times \mathbb{A}$  (see Definition 8.1);*
- (ii) *if  $\varphi(s) \rightarrow -\infty$  as  $s \rightarrow -\infty$ , then the function  $(a, b) \rightarrow c(a, b)$  is  $\mathbb{K}$ -sup-compact on  $\mathbb{A} \times \mathbb{B}$  (see Definition 8.2);*



(iii) if  $\varphi(s) = \varphi_1(s) + \varphi_2(s)$  for each  $s \in \mathbb{R}$ , where  $\varphi_1 : \mathbb{R} \rightarrow \mathbb{R}$  is increasing and  $\varphi_2 : \mathbb{R} \rightarrow \mathbb{R}$  is bounded, then Assumptions (Ma) and (Mb) hold;

(iv) if there exist  $s_* < 0 < s^*$  such that  $\varphi(s_*) > \varphi(s^*)$ , then the game  $\{\mathbb{A}, \mathbb{B}, c\}$  has no solution in pure strategies.

*Proof.* Indeed, due to the symmetric reasonings it is sufficient to verify statements (i,iv) and Assumption (Mb) from statement (iii).

Let us prove statement (i). Consider a sequence  $\{b^{(n)}\}_{n \geq 1}$  that converges to  $b \in \mathbb{B}$  and a sequence  $\{a^{(n)}\}_{n \geq 1} \subset \mathbb{A}$  such that  $\{\varphi(a^{(n)} - b^{(n)})\}_{n \geq 1}$  is bounded above. Note that the sequence  $\{b^{(n)}\}_{n \geq 1} \subset \mathbb{R}_+$  is bounded (above), because it converges. Moreover, the sequence  $\{a^{(n)} - b^{(n)}\}_{n \geq 1}$  is bounded (above), because, on the contrary, if the sequence in hands is unbounded (above), then the continuity of the function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\varphi(s) \rightarrow +\infty$  as  $s \rightarrow +\infty$  yields that the sequence  $\{\varphi(a^{(n)} - b^{(n)})\}_{n \geq 1}$  is unbounded. This is a contradiction. Thus, the sequence  $\{a^{(n)}\}_{n \geq 1} \subset \mathbb{R}_+$  is bounded (above), because the sequence  $\{b^{(n)}\}_{n \geq 1} \subset \mathbb{R}_+$  is bounded (above) and the sequence  $\{a^{(n)} - b^{(n)}\}_{n \geq 1}$  is bounded (above). Therefore,  $\{a^{(n)}\}_{n \geq 1}$  has an accumulation point  $a \in \mathbb{A}$ . To finish the proof of statement (i) we note that  $\varphi(a^{(n)} - b^{(n)}) \rightarrow \varphi(a - b)$  as  $a^{(n)} \rightarrow a$  and  $b^{(n)} \rightarrow b$ , because the function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is continuous; see also Definition 8.1 and Lemma 8.4.

Let us verify Assumption (Mb). Let the function  $\varphi$  possess the representation via the sum of an increasing and a bounded real continuous functions  $\varphi_1$  and  $\varphi_2$  respectively. Then the following equalities and inequalities hold:

$$\begin{aligned} c^b(b) &= \inf_{a \geq 0} (\varphi_1(a - b) + \varphi_2(a - b)) \geq \inf_{a \geq 0} \varphi_1(a - b) + \inf_{a \geq 0} \varphi_2(a - b) \\ &= \varphi_1(-b) + \inf_{a \geq 0} \varphi_2(a - b) = c(0, b) + \inf_{a \geq 0} \varphi_2(a - b) \geq c(0, b) - B, \end{aligned}$$

for each  $b \geq 0$ , where  $B > 0$  be a constant such that  $|\varphi_2(s)| \leq B$  for each  $s \in \mathbb{R}$ . We note that the second equality holds, because the function  $\varphi_1$  is increasing. Therefore,

$$c^-(0, b) \leq \frac{1}{2} c^-(0, b) \leq \frac{1}{2} c^b(b) + \frac{B}{2},$$

for each  $b \geq 0$ , that is, Assumption (Mb) holds.

Let us prove statement (iv). Let there exist  $s_* < 0 < s^*$  such that  $\varphi(s_*) > \varphi(s^*)$ . Then the following inequalities hold:

$$c^b(b) = \inf_{a^* \geq 0} \varphi(a^* - b) \leq \varphi(s^*), \quad c^\sharp(a) = \sup_{b^* \geq 0} \varphi(a - b^*) \geq \varphi(s_*),$$

for each  $a, b \geq 0$ . Therefore,

$$\sup_{b \geq 0} c^b(b) \leq \varphi(s^*) < \varphi(s_*) \leq \inf_{a \geq 0} c^\sharp(a),$$

that is, the game  $\{\mathbb{A}, \mathbb{B}, c\}$  has no solution in pure strategies.  $\square$

Let us establish an example satisfying the conditions of Theorem 3.3.

**Example 7.2.** Let  $\mathbb{A} = \mathbb{B} := \mathbb{R}_+$  and  $c(a, b) := \varphi(a - b)$  for each  $a, b \in \mathbb{R}_+$ , where  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function such that  $\varphi(s) \rightarrow +\infty$  as  $s \rightarrow +\infty$ . Lemma 7.1(i) yields all statements of Theorem 3.3; see also Definition 8.1 and Lemma 8.4.

Let us consider an example satisfying the conditions of Theorem 5.1

**Example 7.3.** Let  $\mathbb{A} = \mathbb{B} := \mathbb{R}_+$  and  $c(a, b) := \varphi(a - b)$  for each  $a, b \in \mathbb{R}_+$ , where  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function such that  $\varphi(s) \rightarrow +\infty$  as  $s \rightarrow +\infty$  and  $\varphi(s) = \varphi_1(s) + \varphi_2(s)$  for each  $s \in \mathbb{R}$ , where  $\varphi_1 : \mathbb{R} \rightarrow \mathbb{R}$  is increasing and  $\varphi_2 : \mathbb{R} \rightarrow \mathbb{R}$  is bounded. Lemma 7.1(i,ii) yields all statements of Theorem 5.1; see also Example 7.2.

Let us consider an example satisfying the conditions of Theorem 6.4.

**Example 7.4.** Let  $\mathbb{A} = \mathbb{B} := \mathbb{R}_+$  and  $c(a, b) := \varphi(a - b)$  for each  $a, b \in \mathbb{R}_+$ , where  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function such that  $\varphi(s) \rightarrow \pm\infty$  as  $s \rightarrow \pm\infty$  and  $\varphi(s) = \varphi_1(s) + \varphi_2(s)$  for each  $s \in \mathbb{R}$ , where  $\varphi_1 : \mathbb{R} \rightarrow \mathbb{R}$  is increasing and  $\varphi_2 : \mathbb{R} \rightarrow \mathbb{R}$  is bounded. Lemma 7.1 yields all statements of Theorem 6.4; see also Example 7.3.

In the following example, the function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is polynomial.

**Example 7.5.** Let  $\mathbb{A} = \mathbb{B} := \mathbb{R}_+$  and  $c(a, b) := \varphi(a - b)$  for each  $a, b \in \mathbb{R}_+$ , where  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a polynomial, that is,  $\varphi(s) = \alpha_M s^M + \alpha_{M-1} s^{M-1} + \dots + \alpha_0$ ,  $s \in \mathbb{R}$ , for some  $\alpha_0, \alpha_1, \dots, \alpha_M \in \mathbb{R}$  such that  $\alpha_M \neq 0$ . Consider the following four alternative cases (a1–a4) and case (b).

(a1): Let  $M$  be even and  $\alpha_M > 0$ . Then condition (v) from Definition 2.3 does not hold because the function  $b \rightarrow c(a, b)$  is not bounded from above for each  $a \geq 0$ . On the other hand, this “game” does not have a solution in pure strategies because

$$\sup_{b \geq 0} \inf_{a \geq 0} \varphi(a - b) \leq \varphi(0) = \alpha_0 < +\infty = \inf_{a \geq 0} \sup_{b \geq 0} \varphi(a - b).$$

Moreover, the “value” (in mixed strategies) for this game equals to  $+\infty$ , that is, the first player has unbounded losses (this does not agree with the notion of equilibrium). Indeed, if we set  $\pi^{\mathbb{B}}(B) := \frac{2}{\pi} \int_B \frac{1}{1+b^2} db$  for each  $B \in \mathcal{B}(\mathbb{B})$ , then

$$\hat{c}(a, \pi^{\mathbb{B}}) = \frac{2}{\pi} \int_{\mathbb{R}_+} \frac{\varphi(a - b)}{1 + b^2} db = +\infty.$$

Therefore,  $\hat{c}^b(\pi^{\mathbb{B}}) = \inf_{a \geq 0} \hat{c}(a, \pi^{\mathbb{B}}) = +\infty$  and  $v_w^b = \sup_{\pi_*^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})} \hat{c}^b(\pi_*^{\mathbb{B}}) \geq \hat{c}^b(\pi^{\mathbb{B}}) = +\infty$  (here we note that  $\mathbb{P}^S(\mathbb{A}) = \mathbb{P}(\mathbb{A})$  and  $\mathbb{P}^S(\mathbb{B}) = \mathbb{P}(\mathbb{B})$ , because the function  $\varphi$  is bounded from below on  $\mathbb{R}$ ; see Remark 4.4). Thus,  $v_w = +\infty$  because  $+\infty \leq v_w^b \leq v_w^\# \leq +\infty$ .

(a2): If  $M$  is even and  $\alpha_M < 0$ , then condition (iv) from Definition 2.3 does not hold, because the function  $a \rightarrow c(a, b)$  is not bounded from below for each  $b \geq 0$ . Moreover (by the symmetric reasonings which follow from case (a1), when  $M$  is even and  $\alpha_M > 0$ ), this “game” does not have a solution in pure strategies and the “value” for this game equals to  $-\infty$ , that is, the second player has unbounded losses (this does not agree with the notion of equilibrium).

(a3): If  $M$  is odd and  $\alpha_M > 0$ , then statements (i–iii) of Lemma 7.1 hold and, therefore, the game  $\{\mathbb{A}, \mathbb{B}, c\}$  satisfies the conditions of Theorem 6.4.

(a4): If  $M$  is odd and  $\alpha_M < 0$ , then conditions (iv,v) from Definition 2.3 do not hold. Moreover, the “lower value” for this game equals to  $-\infty$  and the “upper value” for this game equals to  $+\infty$ , that is, there exists neither “value” nor “solution” for this “game”; see case (a1).

(b): If  $\alpha_1 < 0$ , then statement (iv) of Lemma 7.1 implies that the game  $\{\mathbb{A}, \mathbb{B}, c\}$  has no solution in pure strategies.

## 8 Continuity Properties of Equilibria

In this section we provide sufficient conditions for continuity of the value functions, upper semi-continuity of solution multifunctions, and compactness of solution sets for zero-sum stochastic games with possibly uncountable and noncompact action sets and unbounded payoff functions (Theorems 8.13 and 8.15; see also Theorems 8.16 and 8.17).

Let  $\mathbb{X}$  and  $\mathbb{Y}$  be metric spaces. For a set-valued mapping  $\Phi : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$ , let

$$\text{Dom } \Phi := \{x \in \mathbb{X} : \Phi(x) \neq \emptyset\}.$$

A set-valued mapping  $\Phi : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$  is called *strict* if  $\text{Dom } \Phi = \mathbb{X}$ , that is,  $\Phi : \mathbb{X} \rightarrow S(\mathbb{Y})$  or, equivalently,  $\Phi(x) \neq \emptyset$  for each  $x \in \mathbb{X}$ . For  $Z \subseteq \mathbb{X}$  define the *graph* of a set-valued mapping  $\Phi : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$ , restricted to  $Z$ :

$$\text{Gr}_Z(\Phi) = \{(x, y) \in Z \times \mathbb{Y} : x \in \text{Dom } \Phi, y \in \Phi(x)\}.$$

When  $Z = \mathbb{X}$ , we use the standard notation  $\text{Gr}(\Phi)$  for the graph of  $\Phi : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$  instead of  $\text{Gr}_{\mathbb{X}}(\Phi)$ .

Throughout this section assume that  $\text{Dom } \Phi \neq \emptyset$ . The following definition introduces a notion of a  $\mathbb{K}$ -inf-compact function defined on  $\text{Gr}_{\mathbb{X}}(\Phi)$  for  $\Phi : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$ , while in [9] such functions are defined for  $\Phi : \mathbb{X} \rightarrow S(\mathbb{Y})$ .

**Definition 8.1.** (Feinberg et al. [11, Definition 1.3]) *A function  $f : \text{Gr}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$  is called  $\mathbb{K}$ -inf-compact on  $\text{Gr}(\Phi)$ , if for every  $C \in \mathbb{K}(\text{Dom } \Phi)$  this function is inf-compact on  $\text{Gr}_C(\Phi)$ .*

**Definition 8.2.** (Feinberg et al. [11, Definition 1.4]) *A function  $f : \text{Gr}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$  is called  $\mathbb{K}$ -sup-compact on  $\text{Gr}(\Phi)$  if the function  $-f$  is  $\mathbb{K}$ -inf-compact on  $\text{Gr}(\Phi)$ .*

**Remark 8.3.** According to Remark 2.1, a function  $f : \text{Gr}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$  is  $\mathbb{K}$ -inf-compact /  $\mathbb{K}$ -sup-compact on  $\text{Gr}(\Phi)$  if and only if  $f : \text{Gr}(\Phi) \subseteq \text{Dom } \Phi \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$  is  $\mathbb{K}$ -inf-compact /  $\mathbb{K}$ -sup-compact on  $\text{Gr}(\Phi)$ , where  $\text{Dom } \Phi$  is considered as a metric space with the same metric as on  $\mathbb{X}$ .

The topological meaning of  $\mathbb{K}$ -inf-compactness of a function on a graph of a strict set-valued mapping  $\Phi : \mathbb{X} \rightarrow S(\mathbb{Y})$  is explained in Feinberg et al. [9, Lemma 2.5]; see also Feinberg et al. [6, Lemma 2] and [7, p. 1041]. The following lemma provides necessary and sufficient conditions for  $\mathbb{K}$ -inf-compactness of a function  $f : \text{Gr}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$  for a possibly non-strict set-valued mapping  $\Phi : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$ .

**Lemma 8.4.** (Feinberg et al. [11, Lemma 1.7]) *The function  $f : \text{Gr}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$  is  $\mathbb{K}$ -inf-compact on  $\text{Gr}(\Phi)$  if and only if the following two assumptions hold:*

- (i)  $f : \text{Gr}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$  is lower semi-continuous;
- (ii) if a sequence  $\{x^{(n)}\}_{n=1,2,\dots}$  with values in  $\text{Dom } \Phi$  converges in  $\mathbb{X}$  and its limit  $x$  belongs to  $\text{Dom } \Phi$ , then each sequence  $\{y^{(n)}\}_{n=1,2,\dots}$  with  $y^{(n)} \in \Phi(x^{(n)})$ ,  $n = 1, 2, \dots$ , satisfying the condition that the sequence  $\{f(x^{(n)}, y^{(n)})\}_{n=1,2,\dots}$  is bounded above, has a limit point  $y \in \Phi(x)$ .

The following corollary establishes that assumption (i) in Lemma 8.4 can be substituted by the condition that all the level sets  $\{\mathcal{D}_f(\lambda; \text{Gr}(\Phi))\}_{\lambda \in \mathbb{R}}$  are closed in  $\mathbb{X} \times \mathbb{Y}$ .

**Corollary 8.5.** (Feinberg et al. [11, Corollary 1.8]) *Let  $\Phi : \mathbb{X} \rightarrow S(\mathbb{Y})$  be a strict set-valued mapping and  $f : \text{Gr}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$  be a function satisfying assumption (ii) of Lemma 8.4. Then for each  $\lambda \in \mathbb{R}$  the set  $\mathcal{D}_f(\lambda; \text{Gr}(\Phi))$  is closed in  $\mathbb{X} \times \mathbb{Y}$  if and only if the function  $f : \text{Gr}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$  is lower semi-continuous.*

A set-valued mapping  $F : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$  is *upper semi-continuous* at  $x \in \text{Dom } F$  if, for each neighborhood  $\mathcal{G}$  of the set  $F(x)$ , there is a neighborhood of  $x$ , say  $U(x)$ , such that  $F(x^*) \subseteq \mathcal{G}$  for all  $x^* \in U(x) \cap \text{Dom } F$ ; a set-valued mapping  $F : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$  is *lower semi-continuous* at  $x \in \text{Dom } F$  if, for each open set  $\mathcal{G}$  with  $F(x) \cap \mathcal{G} \neq \emptyset$ , there is a neighborhood of  $x$ , say  $U(x)$ , such that if  $x^* \in U(x) \cap \text{Dom } F$ , then  $F(x^*) \cap \mathcal{G} \neq \emptyset$  (see e.g., Berge [3, p. 109] or Zgurovsky et al. [22, Chapter 1, p. 7]). A set-valued mapping is called *upper / lower semi-continuous*, if it is upper / lower semi-continuous at all  $x \in \text{Dom } F$ . A set-valued mapping  $F : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$  is  $\mathbb{K}$ -*upper semi-compact* if for each  $C \in \mathbb{K}(\text{Dom } \Phi)$  the set  $\text{Gr}_C(F)$  is compact; see e.g. Feinberg et al. [7, Definition 2.3].

The sufficient conditions for  $\mathbb{K}$ -inf-compactness are the following.

**Lemma 8.6.** (Feinberg et al. [11, Lemma 1.9]) *Let  $\Phi : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$  be a set-valued mapping and  $f : \text{Gr}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$  be a function. Then the following statements hold:*

- (a) *if  $f : \text{Gr}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$  is inf-compact on  $\text{Gr}(\Phi)$ , then the function  $f$  is  $\mathbb{K}$ -inf-compact on  $\text{Gr}(\Phi)$ ;*
- (b) *if  $f : \text{Gr}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$  is lower semi-continuous and  $\Phi : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$  is upper semi-continuous and compact-valued at each  $x \in \text{Dom } \Phi$ , then the function  $f$  is  $\mathbb{K}$ -inf-compact on  $\text{Gr}(\Phi)$ .*

The following lemma provides the necessary and sufficient conditions for  $\mathbb{K}$ -upper semi-compactness of a possibly non-strict set-valued mapping  $\Phi : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$ .

**Lemma 8.7.** (Feinberg et al. [11, Lemma 1.10]) *A set-valued mapping  $\Phi : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$  is  $\mathbb{K}$ -upper semi-compact if and only if it is upper semi-continuous and compact-valued at each  $x \in \text{Dom } \Phi$ .*

Define families of games with action sets and payoff functions depending on a parameter. Let  $\mathbb{X}, \mathbb{A}$  and  $\mathbb{B}$  be Borel subsets of Polish spaces,  $K_{\mathbb{A}} \in \mathcal{B}(\mathbb{X} \times \mathbb{A})$ , where  $\mathcal{B}(\mathbb{X} \times \mathbb{A}) = \mathcal{B}(\mathbb{X}) \otimes \mathcal{B}(\mathbb{A})$ ,  $K_{\mathbb{B}} \in \mathcal{B}(\mathbb{X} \times \mathbb{B})$ , where  $\mathcal{B}(\mathbb{X} \times \mathbb{B}) = \mathcal{B}(\mathbb{X}) \otimes \mathcal{B}(\mathbb{B})$ . It is assumed that for each  $x \in \mathbb{X}$  the sets  $K_{\mathbb{A}}$  and  $K_{\mathbb{B}}$  satisfy the following two conditions:

$$A(x) := \{a \in \mathbb{A} : (x, a) \in K_{\mathbb{A}}\} \neq \emptyset \quad \text{and} \quad B(x) := \{b \in \mathbb{B} : (x, b) \in K_{\mathbb{B}}\} \neq \emptyset.$$

Let

$$\mathcal{K} := \{(x, a, b) \in \mathbb{X} \times \mathbb{A} \times \mathbb{B} : x \in \mathbb{X}, a \in A(x), b \in B(x)\}.$$

**Remark 8.8.** We note that  $\text{Gr}(A) = K_{\mathbb{A}}$ ,  $\text{Gr}(B) = K_{\mathbb{B}}$ , and  $\mathcal{K} = \text{Gr}(A \times B)$ , where  $(A \times B)(x) := \{(a, b) : a \in A(x), b \in B(x)\}$ ,  $x \in \mathbb{X}$ . We note also that  $\mathcal{K} = \text{Gr}(\tilde{B})$ , where  $\tilde{B}(x, a) := B(x)$ ,  $(x, a) \in K_{\mathbb{A}}$ . If we set  $\tilde{A}(x, b) := A(x)$ ,  $(x, b) \in K_{\mathbb{B}}$ , then  $\text{Gr}(\tilde{A}) = \{(x, b, a) : (x, a, b) \in \mathcal{K}\}$  and  $\mathcal{K} = \{(x, a, b) : (x, b, a) \in \text{Gr}(\tilde{A})\}$ .

Consider the family of two-person zero-sum games

$$\{\{A(x), B(x), c(x, \cdot, \cdot)\} : x \in \mathbb{X}\}.$$

satisfying for each  $x \in \mathbb{X}$  all the assumptions from Definition 2.3. Define the function  $c^{\mathbb{A} \leftrightarrow \mathbb{B}} : \text{Gr}(\tilde{A}) \subseteq (\mathbb{X} \times \mathbb{B}) \times \mathbb{A} \rightarrow \overline{\mathbb{R}}$ ,

$$c^{\mathbb{A} \leftrightarrow \mathbb{B}}(x, b, a) := c(x, a, b), \quad (x, a, b) \in \mathcal{K}. \quad (8.1)$$

In this subsection we consider the following assumptions:

- (A1) the function  $c^{\mathbb{A} \leftrightarrow \mathbb{B}} : \text{Gr}(\tilde{A}) \subseteq (\mathbb{X} \times \mathbb{B}) \times \mathbb{A} \rightarrow \mathbb{R} \cup \{+\infty\}$  defined in (8.1) is  $\mathbb{K}$ -inf-compact on  $\text{Gr}(\tilde{A})$ ;
- (A2) the function  $c : \mathcal{K} \subseteq (\mathbb{X} \times \mathbb{A}) \times \mathbb{B} \rightarrow \mathbb{R} \cup \{-\infty\}$  is  $\mathbb{K}$ -sup-compact on  $\mathcal{K}$ ;
- (A3)  $A : \mathbb{X} \rightarrow S(\mathbb{A})$  is a lower semi-continuous set-valued mapping;
- (A4)  $B : \mathbb{X} \rightarrow S(\mathbb{B})$  is a lower semi-continuous set-valued mapping.

**Remark 8.9.** According to Lemma 8.4 and Remark 8.8, Assumption (A1) holds if and only if the following two conditions hold:

- (i) the mapping  $c : \mathcal{K} \subset \mathbb{X} \times \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{R} \cup \{+\infty\}$  is lower semi-continuous;
- (ii) if a sequence  $\{x^{(n)}, b^{(n)}\}_{n=1,2,\dots}$  with values in  $K_{\mathbb{B}}$  converges and its limit  $(x, b)$  belongs to  $K_{\mathbb{B}}$ , then each sequence  $\{a^{(n)}\}_{n=1,2,\dots}$  with  $(x^{(n)}, a^{(n)}, b^{(n)}) \in \mathcal{K}$ ,  $n = 1, 2, \dots$ , satisfying the condition that the sequence  $\{c(x^{(n)}, a^{(n)}, b^{(n)})\}_{n=1,2,\dots}$  is bounded above, has a limit point  $a \in A(x)$ .

**Remark 8.10.** According to Lemma 8.4 and Remark 8.8, Assumption (A2) holds if and only if the following two conditions hold:

- (i) the mapping  $c : \mathcal{K} \subset \mathbb{X} \times \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{R} \cup \{-\infty\}$  is upper semi-continuous;
- (ii) if a sequence  $\{x^{(n)}, a^{(n)}\}_{n=1,2,\dots}$  with values in  $K_{\mathbb{A}}$  converges and its limit  $(x, a)$  belongs to  $K_{\mathbb{A}}$ , then each sequence  $\{b^{(n)}\}_{n=1,2,\dots}$  with  $(x^{(n)}, a^{(n)}, b^{(n)}) \in \mathcal{K}$ ,  $n = 1, 2, \dots$ , satisfying the condition that the sequence  $\{c(x^{(n)}, a^{(n)}, b^{(n)})\}_{n=1,2,\dots}$  is bounded from below, has a limit point  $b \in B(x)$ .

**Remark 8.11.** Assumptions (A1) and (A2) imply that the payoff from Player I to Player II,  $-\infty < c(x, a, b) < +\infty$ , for choosing actions  $a \in A(x)$  and  $b \in B(x)$  in a state  $x \in \mathbb{X}$ , is a continuous function.

**Remark 8.12.** Let  $\{\{A(x), B(x), c(x, \cdot, \cdot)\} : x \in \mathbb{X}\}$  be the family of two-person zero-sum games. Further let  $\hat{c}^{\sharp}(x)$  and  $\hat{c}^{\flat}(x)$  be defined in (2.1) and  $v_w^{\flat}(x)$ ,  $v_w^{\sharp}(x)$ ,  $v_w(x)$ , and  $v(x)$  denote the lower value, the upper value, the weak value, and the value of the game  $\{A(x), B(x), c(x, \cdot, \cdot)\}$  if they exist,  $x \in \mathbb{X}$ .

The following theorem provides the sufficient conditions of the lower semi-continuity for the weak value of a family of two-person zero-sum games with possibly noncompact action sets and unbounded payoffs.

**Theorem 8.13.** *Let the family of two-person zero-sum games  $\{ \{A(x), B(x), c(x, \cdot, \cdot) \} : x \in \mathbb{X} \}$  satisfy Assumptions (A1, A4) and (MbW). Then the following statements hold:*

- (i) *for each  $x \in \mathbb{X}$  the game  $\{A(x), B(x), c(x, \cdot, \cdot)\}$  has the weak value  $v_w(x)$ , that is, the following equalities hold:*

$$v_w^b(x) = \sup_{\pi^B \in \mathbb{P}(B(x))} \hat{c}^b(x, \pi^B) = \inf_{\pi^A \in \mathbb{P}(A(x))} \hat{c}^A(x, \pi^A) = v_w^A(x) = v_w(x); \quad (8.2)$$

*moreover,  $v_w : \mathbb{X} \rightarrow \mathbb{R}$  is a lower semi-continuous function;*

- (ii) *the sets  $\{\mathbb{P}_{v_w(x)}^\#(A(x)) : x \in \mathbb{X}\}$  satisfy the following properties:*

- (a) *for each  $x \in \mathbb{X}$  the set  $\mathbb{P}_{v_w(x)}^\#(A(x))$  is a nonempty convex compact subset of  $\mathbb{P}(\mathbb{A})$ ;*
- (b) *the graph  $\text{Gr}(\mathbb{P}_{v_w(\cdot)}^\#(A(\cdot))) = \{(x, \pi^A) : x \in \mathbb{X}, \pi^A \in \mathbb{P}_{v_w(x)}^\#(A(x))\}$  is a Borel subset of  $\mathbb{X} \times \mathbb{P}(\mathbb{A})$ ;*
- (c) *there exists a measurable mapping  $\phi^A : \mathbb{X} \rightarrow \mathbb{P}(\mathbb{A})$  such that  $\phi^A(x) \in \mathbb{P}_{v_w(x)}^\#(A(x))$  for each  $x \in \mathbb{X}$ .*

*Proof.* Assumption (A1) and Feinberg et al. [11, Corollary 4.3], being applied to  $\mathbb{X} := \mathbb{X} \times \mathbb{B}$ ,  $\mathbb{Y} := \mathbb{A}$ ,  $f := c^{\mathbb{A} \leftrightarrow \mathbb{B}}$  on  $\text{Gr}(\tilde{A})$ , and  $f := +\infty$  on the complement of  $\text{Gr}(\tilde{A})$ , yield that the mapping  $\hat{c}^{\mathbb{A} \leftrightarrow \mathbb{B}} : \text{Gr}(\mathbb{P}(\tilde{A}(\cdot, \cdot))) \subseteq (\mathbb{X} \times \mathbb{B}) \times \mathbb{P}(\mathbb{A}) \rightarrow \mathbb{R} \cup \{+\infty\}$ , where

$$\hat{c}^{\mathbb{A} \leftrightarrow \mathbb{B}}(x, b, \pi^A) = \int_{A(x)} c(x, a, b) \pi^A(da), \quad (x, b) \in \mathbb{K}_B, \pi^A \in \mathbb{P}(\tilde{A}(x, b)), \quad (8.3)$$

is  $\mathbb{K}$ -inf-compact on  $\text{Gr}(\mathbb{P}(\tilde{A}(\cdot, \cdot)))$ . Equalities (8.2) follow from Theorem 5.1. The least statements follow from Feinberg et al. [11, Theorem 3.6], being applied to  $\mathbb{X} := \mathbb{X}$ ,  $\mathbb{A} := \mathbb{P}(\mathbb{A})$ ,  $\mathbb{B} := \mathbb{B}$ ,  $\Phi_A(\cdot) := \mathbb{P}(A(\cdot))$ ,  $\Phi_B(x, \pi^A) := B(x)$ ,  $x \in \mathbb{X}$  and  $\pi^A \in \mathbb{P}(A(x))$ , and  $f(x, \pi^A, b) := \hat{c}(x, \pi^A, b)$ ,  $(x, \pi^A, b) \in \{(x, \pi^A, b) \in \mathbb{X} \times \mathbb{P}(\mathbb{A}) \times \mathbb{B} : (x, b) \in \mathbb{K}_B, \pi^A \in \mathbb{P}(A(x))\}$ , and from Feinberg et al. [9, Theorem 3.3].  $\square$

The following example describes a family of two-person zero-sum games satisfying Assumptions (A1), (A4) and (MbW). Payoff functions are unbounded and decision sets are noncompact for the games in this family.

**Example 8.14.** Let  $\mathbb{X} = \mathbb{A} = \mathbb{B} = \mathbb{R}$ ,  $K_A = K_B = \mathbb{R}^2$ ,  $K = \mathbb{R}^3$ ,  $c(x, a, b) = \varphi_X(x) + \varphi_A(a) + \varphi_B(b)$ ,  $(x, a, b) \in K$ , where  $\varphi_X, \varphi_A, \varphi_B : \mathbb{R} \rightarrow \mathbb{R}$  are continuous functions such that  $\varphi_A(a) \rightarrow +\infty$  as  $|a| \rightarrow \infty$ . Then  $c$  is a continuous function on  $\mathbb{R}^3$  and it satisfies Assumption (A1). Indeed, let a sequence  $\{x^{(n)}, b^{(n)}\}_{n=1,2,\dots}$  with values in  $\mathbb{R}^2$  converges and its limit  $(x, b)$  belongs to  $\mathbb{R}^2$ , a sequence  $\{a^{(n)}\}_{n=1,2,\dots}$  with  $(x^{(n)}, a^{(n)}, b^{(n)}) \in \mathbb{R}^3$ ,  $n = 1, 2, \dots$ , satisfy the condition that the sequence  $\{c(x^{(n)}, a^{(n)}, b^{(n)})\}_{n=1,2,\dots}$  is bounded above. Then the sequence  $\{\varphi_A(a^{(n)})\}_{n=1,2,\dots}$  is bounded above and, since  $\varphi_A(a) \rightarrow +\infty$  as  $|a| \rightarrow \infty$ , then the sequence  $\{a^{(n)}\}_{n=1,2,\dots}$  has a limit point  $a \in A(x) = \mathbb{R}$ . Therefore, Assumption (A1) holds. Assumption (A4) holds, because the multi-valued mapping  $\Phi : \mathbb{R} \rightarrow S(\mathbb{R})$ ,  $\Phi(s) = \mathbb{R}$ ,  $s \in \mathbb{R}$ , is lower semi-continuous on  $\mathbb{R}$ . Assumption (MbW) is verified as in Example 4.6.



The following theorem and its corollary describes sufficient conditions for the continuity of the value function and upper semi-continuity of the solution multifunctions for a family of two-person zero-sum games with possibly noncompact action sets and unbounded payoffs.

**Theorem 8.15.** *Let a family of two-person zero-sum games  $\{A(x), B(x), c(x, \cdot, \cdot)\} : x \in \mathbb{X}\}$  satisfy Assumptions (A1–A4), (MaW) and (MbW). Then the following statements hold:*

- (i) *for each  $x \in \mathbb{X}$  the game  $\{A(x), B(x), c(x, \cdot, \cdot)\}$  has the value  $v(x)$  and the solution  $(\pi^A, \pi^B) \in \mathbb{P}_{v(x)}^{\sharp}(A(x)) \times \mathbb{P}_{v(x)}^b(B(x))$  in safe strategies. Moreover,  $\mathbb{P}_{v(x)}^{\sharp}(A(x)) \subset \mathbb{P}^S(A(x))$  and  $\mathbb{P}_{v(x)}^b(B(x)) \subset \mathbb{P}^S(B(x))$  for each  $x \in \mathbb{X}$ , and  $v : \mathbb{X} \rightarrow \mathbb{R}$  is a continuous function;*
- (ii) *the sets  $\{\mathbb{P}_{v(x)}^{\sharp}(A(x)) : x \in \mathbb{X}\}$  satisfy the following properties:*
  - (a) *for each  $x \in \mathbb{X}$  the set  $\mathbb{P}_{v(x)}^{\sharp}(A(x))$  is a nonempty convex compact subset of  $\mathbb{P}(\mathbb{A})$ ;*
  - (b) *the multifunction  $\mathbb{P}_{v(\cdot)}^{\sharp}(A(\cdot)) : \mathbb{X} \rightarrow \mathbb{K}(\mathbb{P}(\mathbb{A}))$  is upper semi-continuous;*
- (iii) *the sets  $\{\mathbb{P}_{v(x)}^b(B(x)) : x \in \mathbb{X}\}$  satisfy the following properties:*
  - (a) *for each  $x \in \mathbb{X}$  the set  $\mathbb{P}_{v(x)}^b(B(x))$  is a nonempty convex compact subset of  $\mathbb{P}(\mathbb{B})$ ;*
  - (b) *the multifunction  $\mathbb{P}_{v(\cdot)}^b(B(\cdot)) : \mathbb{X} \rightarrow \mathbb{K}(\mathbb{P}(\mathbb{B}))$  is upper semi-continuous;*
- (iv) *there exist measurable mappings  $\phi^A : \mathbb{X} \rightarrow \mathbb{P}(\mathbb{A})$  and  $\phi^B : \mathbb{X} \rightarrow \mathbb{P}(\mathbb{B})$  such that  $\phi^A(x) \in \mathbb{P}_{v(x)}^{\sharp}(A(x))$  and  $\phi^B(x) \in \mathbb{P}_{v(x)}^b(B(x))$  for all  $x \in \mathbb{X}$ . Moreover, for each  $x \in \mathbb{X}$  a pair of strategies  $(\pi^A(x), \pi^B(x)) \in \mathbb{P}^S(A(x)) \times \mathbb{P}^S(B(x))$  is a solution of the game  $\{A(x), B(x), c(x, \cdot, \cdot)\}$  in safe strategies if and only if  $\pi^A(x) \in \mathbb{P}_{v(x)}^{\sharp}(A(x))$  and  $\pi^B(x) \in \mathbb{P}_{v(x)}^b(B(x))$ .*

*Proof.* According to Theorem 6.4, Theorem 8.13, being applied twice to  $\{A(x), B(x), c(x, \cdot, \cdot)\} : x \in \mathbb{X}\}$  and  $\{B(x), A(x), -c^{\mathbb{A} \leftrightarrow \mathbb{B}}(x, \cdot, \cdot)\} : x \in \mathbb{X}\}$ , where  $c^{\mathbb{A} \leftrightarrow \mathbb{B}}(x, b, a) := c(x, a, b)$  for each  $x \in \mathbb{X}$ ,  $a \in A(x)$  and  $b \in B(x)$ , yield statements (i), (ii)(a) and (iii)(a). Statement (ii)(b) follows from Feinberg et al. [11, Theorem 3.9], being applied to  $\mathbb{X} := \mathbb{X}$ ,  $\mathbb{A} := \mathbb{P}(\mathbb{A})$ ,  $\mathbb{B} := \mathbb{P}(\mathbb{B})$ ,  $\Phi_A(\cdot) := \mathbb{P}(A(\cdot))$ ,  $\Phi_B(x, \pi^A) := B(x)$ ,  $x \in \mathbb{X}$  and  $\pi^A \in \mathbb{P}(A(x))$ , and  $f(x, \pi^A, b) := \hat{c}(x, \pi^A, b)$ ,  $(x, \pi^A, b) \in \{(x, \pi^A, b) \in \mathbb{X} \times \mathbb{P}(\mathbb{A}) \times \mathbb{B} : (x, b) \in \mathbb{K}_{\mathbb{B}}, \pi^A \in \mathbb{P}(A(x))\}$ , because the mapping  $\hat{c}^{\mathbb{A} \leftrightarrow \mathbb{B}} : \text{Gr}(\mathbb{P}(\tilde{A}(\cdot, \cdot))) \subseteq (\mathbb{X} \times \mathbb{B}) \times \mathbb{P}(\mathbb{A}) \rightarrow \mathbb{R} \cup \{+\infty\}$  defined in (8.3) is  $\mathbb{K}$ -inf-compact on  $\text{Gr}(\mathbb{P}(\tilde{A}(\cdot, \cdot)))$ ; see the proof of Theorem 8.13. Statement (iii)(b) follows from statement (ii)(b) of Theorem 8.15, being applied to  $\{B(x), A(x), -c^{\mathbb{A} \leftrightarrow \mathbb{B}}(x, \cdot, \cdot)\} : x \in \mathbb{X}\}$ . Statement (iv) directly follow from statements (ii,iii), Theorem 6.4(iii) and Kuratowski and Ryll-Nardzewski measurable selection theorem.  $\square$

The following two theorems are versions of Theorems 8.13 and 8.15 respectively, when a family of two-person zero-sum games  $\{A(x), B(x), c(x, \cdot, \cdot)\} : x \in \mathbb{X}\}$  satisfies the following additional assumption:

**(Co)** let the following conditions hold:

- (Co1)** the sets  $\mathbb{A}$  and  $\mathbb{B}$  are convex subsets of vector spaces and the sets  $A(x)$  and  $B(x)$  are convex for each  $x \in \mathbb{X}$ ;

(Co2) the function  $a \rightarrow c(x, a, b)$  is convex on  $A(x)$  for each  $x \in \mathbb{X}$  and  $b \in B(x)$ ;

(Co3) the function  $b \rightarrow c(x, a, b)$  is concave on  $B(x)$  for each  $x \in \mathbb{X}$  and  $a \in A(x)$ .

Note that under Assumptions (Co) we may omit Assumptions (MaW, MbW) in Theorems 8.13 and 8.15.

**Theorem 8.16.** *Let the family of two-person zero-sum games  $\{\{A(x), B(x), c(x, \cdot, \cdot)\} : x \in \mathbb{X}\}$  satisfy Assumptions (A1, A4) and (Co). Then the following statements hold:*

(i) *for each  $x \in \mathbb{X}$  the game  $\{A(x), B(x), c(x, \cdot, \cdot)\}$  has a value  $v(x)$  in pure strategies, that is, the following equalities hold:*

$$\begin{aligned} v^b(x) &= \sup_{b \in B(x)} c^b(x, b) = \sup_{b \in B(x)} \inf_{a \in A(x)} c(x, a, b) \\ &= \inf_{a \in A(x)} \sup_{b \in B(x)} c(x, a, b) = \inf_{a \in A(x)} c^\sharp(x, a) = v^\sharp(x) = v(x); \end{aligned} \quad (8.4)$$

moreover,  $v : \mathbb{X} \rightarrow \mathbb{R}$  is a lower semi-continuous function;

(ii) *the sets  $\{A_{v(x)}^\sharp(x) := \{a \in \mathbb{A} : c^\sharp(x, a) = v^\sharp(x)\} : x \in \mathbb{X}\}$  satisfy the following properties:*

(a) *for each  $x \in \mathbb{X}$  the set  $A_{v(x)}^\sharp(x)$  is a nonempty convex compact subset of  $\mathbb{A}$ ;*

(b) *the graph  $\text{Gr}(A_{v(\cdot)}^\sharp(\cdot)) = \{(x, a) : x \in \mathbb{X}, a \in A_{v(x)}^\sharp(x)\}$  is a Borel subset of  $\mathbb{X} \times \mathbb{A}$ ;*

(c) *there exists a measurable mapping  $a(\cdot) : \mathbb{X} \rightarrow \mathbb{A}$  such that  $a(x) \in A_{v(x)}^\sharp(x)$  for each  $x \in \mathbb{X}$ .*

*Proof.* Let us fix an arbitrary  $x \in \mathbb{X}$ . Assumption (Co) imply that

$$\sup_{\pi^\mathbb{B} \in \mathbb{P}^{fs}(B(x))} \hat{c}(x, \pi^\mathbb{B}) = \sup_{b \in B(x)} \inf_{a \in A(x)} c(x, a, b). \quad (8.5)$$

Indeed, the inequality

$$\sup_{b \in B(x)} \inf_{a \in A(x)} c(x, a, b) \leq \sup_{\pi^\mathbb{B} \in \mathbb{P}^{fs}(B(x))} \inf_{a \in A(x)} \hat{c}(x, a, \pi^\mathbb{B}). \quad (8.6)$$

holds, because each pure strategy  $b \in B(x)$  for Player II can be interpreted as a mixed one  $\pi^\mathbb{B} \in \mathbb{P}(B(x))$  concentrated in a point  $b$ . To verify the converse inequality:

$$\sup_{\pi^\mathbb{B} \in \mathbb{P}^{fs}(B(x))} \inf_{a \in A(x)} \hat{c}(x, a, \pi^\mathbb{B}) \leq \sup_{b \in B(x)} \inf_{a \in A(x)} c(x, a, b), \quad (8.7)$$

we note that for each  $\pi^\mathbb{B} \in \mathbb{P}^{fs}(B(x))$  there exist  $K = 1, 2, \dots$ ,  $\{\alpha^{(1)}, \dots, \alpha^{(K)}\} \subset \mathbb{R}_+$  and  $\{b^{(1)}, \dots, b^{(K)}\} \subset B(x)$  such that  $\alpha^{(1)} + \dots + \alpha^{(K)} = 1$  and  $\pi^\mathbb{B}(B) = \alpha^{(1)} \mathbf{I}\{b^{(1)} \in B\} + \dots + \alpha^{(K)} \mathbf{I}\{b^{(K)} \in B\}$  for each  $B \in \mathcal{B}(B(x))$ , where  $\mathbf{I}\{b^{(i)} \in B\} = 1$  when  $b^{(i)} \in B$  and  $\mathbf{I}\{b^{(i)} \in B\} = 0$  otherwise;  $i = 1, \dots, K$ . Therefore,

$$\hat{c}(x, a, \pi^\mathbb{B}) = \alpha^{(1)} c(x, a, b^{(1)}) + \dots + \alpha^{(K)} c(x, a, b^{(K)}) \leq c(x, a, b(\pi^\mathbb{B})), \quad (8.8)$$

for each  $a \in A(x)$  and  $\pi^\mathbb{B} \in \mathbb{P}^{fs}(B(x))$ , where  $b(\pi^\mathbb{B}) := \alpha^{(1)} b^{(1)} + \dots + \alpha^{(K)} b^{(K)}$  and the inequality in (8.8) holds, because the function  $b^* \rightarrow c(x, a, b^*)$  is concave on  $B(x)$ . Inequality (8.8) yields that

$$\inf_{a \in A(x)} \hat{c}(x, a, \pi^\mathbb{B}) \leq \inf_{a \in A(x)} c(x, a, b(\pi^\mathbb{B})) \leq \sup_{b^* \in B(x)} \inf_{a \in A(x)} c(x, a, b^*)$$

for each  $\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(B(x))$ , that is, inequality (8.7) holds. The definition of  $\hat{c}^b$  and inequalities (8.6) and (8.7) imply equality (8.5).

Let us prove statement (ii)(a). Fix an arbitrary  $x \in \mathbb{X}$ . Equality (8.5) and Theorem 3.3 yield that

$$\sup_{b \in B(x)} \inf_{a \in A(x)} c(x, a, b) = \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(B(x))} \hat{c}^b(x, \pi^{\mathbb{B}}) = \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^S(B(x))} \hat{c}^b(x, \pi^{\mathbb{B}}) = \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(A(x))} \hat{c}^{\#}(x, \pi^{\mathbb{A}}). \quad (8.9)$$

Thus, Aubin and Ekeland [2, Theorem 6.2.7] implies that

$$\inf_{\pi^{\mathbb{A}} \in \mathbb{P}(A(x))} \hat{c}^{\#}(x, \pi^{\mathbb{A}}) = \inf_{a \in A(x)} \sup_{b \in B(x)} c(x, a, b). \quad (8.10)$$

and the set  $A_{v(x)}^{\#}(x)$  is a nonempty convex compact subset of  $\mathbb{A}$ , that is, statement (ii)(a) holds.

Let us prove statement (i). For each  $x \in \mathbb{X}$  equalities (8.9) and (8.10) imply equalities (8.4), that is, the game  $\{A(x), B(x), c(x, \cdot, \cdot)\}$  has a value  $v(x)$  in pure strategies. Moreover, Theorem 8.13(i) yields that the function  $v : \mathbb{X} \rightarrow \mathbb{R}$  is lower semi-continuous. Therefore, statement (i) holds.

Statements (ii)(b,c) follow from Feinberg et al. [9, Theorems 3.1 and 3.3] (see also Feinberg et al. [7]), being applied to  $X := \mathbb{X}$ ,  $Y := \mathbb{A}$ ,  $\Phi := A$ , and  $u := c^{\#}$ . Here we note that  $\mathbb{K}$ -inf-compactness of  $c^{\#}$  on  $\text{Gr}(A)$  follows from Assumptions (A1,A4) and Feinberg et al. [11, Theorem 3.2], being applied to  $\mathbb{X} := \mathbb{X}$ ,  $\mathbb{A} := \mathbb{A}$ ,  $\mathbb{B} := \mathbb{B}$ ,  $\Phi_{\mathbb{A}} := A$ ,  $\Phi_{\mathbb{B}}(x, a) := B(x)$ ,  $x \in \mathbb{X}$ ,  $a \in A(x)$ , and  $f := c$ .  $\square$

**Theorem 8.17.** *Let a family of two-person zero-sum games  $\{A(x), B(x), c(x, \cdot, \cdot)\} : x \in \mathbb{X}$  satisfy Assumptions (A1–A4) and (Co). Then the following statements hold:*

- (i) *for each  $x \in \mathbb{X}$  the game  $\{A(x), B(x), c(x, \cdot, \cdot)\}$  has a solution  $(a, b) \in A(x) \times B(x)$  in pure strategies. Moreover, the value function  $v : \mathbb{X} \rightarrow \mathbb{R}$  defined in (8.4) is a continuous function;*
- (ii) *the sets  $\{A_{v(x)}^{\#}(x) := \{a \in \mathbb{A} : c^{\#}(x, a) = v^{\#}(x)\} : x \in \mathbb{X}\}$  satisfy the following properties:*
  - (a) *for each  $x \in \mathbb{X}$  the set  $A_{v(x)}^{\#}(x)$  is a nonempty convex compact subset of  $\mathbb{A}$ ;*
  - (b) *the multifunction  $A_{v(\cdot)}^{\#}(\cdot) : \mathbb{X} \rightarrow \mathbb{K}(\mathbb{A})$  is upper semi-continuous;*
- (iii) *the sets  $\{B_{v(x)}^b(x) := \{b \in \mathbb{B} : c^b(x, b) = v^b(x)\} : x \in \mathbb{X}\}$  satisfy the following properties:*
  - (a) *for each  $x \in \mathbb{X}$  the set  $B_{v(x)}^b(x)$  is a nonempty convex compact subset of  $\mathbb{B}$ ;*
  - (b) *the multifunction  $B_{v(\cdot)}^b(\cdot) : \mathbb{X} \rightarrow \mathbb{K}(\mathbb{B})$  is upper semi-continuous.*
- (iv) *there exist measurable mappings  $a(\cdot) : \mathbb{X} \rightarrow \mathbb{A}$  and  $b(\cdot) : \mathbb{X} \rightarrow \mathbb{B}$  such that  $a(x) \in A_{v(x)}^{\#}(x)$  and  $b(x) \in B_{v(x)}^b(x)$  for all  $x \in \mathbb{X}$ . Moreover, for each  $x \in \mathbb{X}$  a pair of strategies  $(a(x), b(x)) \in A(x) \times B(x)$  is a solution of the game  $\{A(x), B(x), c(x, \cdot, \cdot)\}$  in pure strategies if and only if  $a(x) \in A_{v(x)}^{\#}(x)$  and  $b(x) \in B_{v(x)}^b(x)$ .*

*Proof.* Statements (i), (ii)(a) and (iii)(a) follow from Theorem 8.16, being applied twice to  $\{A(x), B(x), c(x, \cdot, \cdot)\} : x \in \mathbb{X}$  and  $\{B(x), A(x), -c^{\mathbb{A} \leftrightarrow \mathbb{B}}(x, \cdot, \cdot)\} : x \in \mathbb{X}$ , where  $c^{\mathbb{A} \leftrightarrow \mathbb{B}}(x, b, a) := c(x, a, b)$  for each  $x \in \mathbb{X}$ ,  $a \in A(x)$  and  $b \in B(x)$ . Statement (ii)(b) follows from Feinberg et al. [11, Theorem 3.9],

being applied to  $X := \mathbb{X}$ ,  $A := \mathbb{A}$ ,  $B := \mathbb{B}$ ,  $\Phi_A := A(\cdot)$ ,  $\Phi_B(x, a) := B(x)$ ,  $x \in \mathbb{X}$  and  $a \in A(x)$ , and  $f := c$ . Statement (iii)(b) follows from statement (ii)(b), being applied to  $\{\{B(x), A(x), -c^{\mathbb{A} \leftrightarrow \mathbb{B}}(x, \cdot, \cdot)\} : x \in \mathbb{X}\}$ . Statement (iv) directly follow from statements (ii) and (iii) and Kuratowski and Ryll-Nardzewski measurable selection theorem; see also Aubin and Ekeland [2, Proposition 1, Chapter 6].  $\square$

## Appendix A Relations of Theorem 3.3 to Previously Known Results

The lopsided minimax theorem for convex possibly noncompact sets of actions in the standard Borel spaces has the following formulation for the case when  $\mathbb{A}$  is a Borel subset of a Polish space.

**Theorem A.18.** (cp. Aubin and Ekeland [2, Theorem 6.2.7]) *Let a two-person zero-sum game  $\{\mathbb{A}, \mathbb{B}, c\}$  introduced in Definition 2.3 satisfies conditions (i) and (ii) of Theorem 3.3; the spaces  $\mathbb{A}$  and  $\mathbb{B}$  are convex subsets of vector spaces; the function  $a \rightarrow c(a, b)$  is convex for each  $b \in \mathbb{B}$ ; and the function  $b \rightarrow c(a, b)$  is concave for each  $a \in \mathbb{A}$ . Then*

$$\inf_{a \in \mathbb{A}} \sup_{b \in \mathbb{B}} c(a, b) = \sup_{b \in \mathbb{B}} \inf_{a \in \mathbb{A}} c(a, b) =: V.$$

Moreover, there exists  $a^* \in \mathbb{A}$  such that  $c^\sharp(a^*) = V$ .

**Remark A.19.** Theorem A.18 is the corollary from Theorem 3.3. Indeed, if we repeat several lines from the proof of Theorem 8.16, we obtain all statements of Theorem A.18. Moreover,

$$\sup_{b \in \mathbb{B}} \inf_{a \in \mathbb{A}} c(a, b) = \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(\mathbb{B})} \hat{c}^\flat(\pi^{\mathbb{B}}) = \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^{\mathbb{A}}) = \inf_{a \in \mathbb{A}} \sup_{b \in \mathbb{B}} c(a, b).$$

Therefore, Theorem A.18 is the corollary from Theorem 3.3.

**Remark A.20.** Fan [4, Theorem 2]; Sion [20, Theorems 3.4 and 4.2] (see also Kneser [15]); Mertens et al. [17, Theorem I.1.1] are corollaries from Theorem A.18, when at least one of the decision sets is assumed to be compact.

The minimax theorem for nonconvex decision sets and for a standard Borel space  $\mathbb{A}$  has the following formulation.

**Theorem A.21.** (cp. Mertens et al. [17, Proposition I.1.9]) *Let  $\mathbb{A}$  be a metrizable compact topological space (compact metric space),  $\mathbb{B}$  be a nonempty set. Assume that, for each  $b \in \mathbb{B}$ ,  $c(\cdot, b)$  is lower semi-continuous. Then*

$$\min_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(\mathbb{B})} \int_{\mathbb{A}} \int_{\mathbb{B}} c(a, b) \pi^{\mathbb{B}}(db) \pi^{\mathbb{A}}(da) = \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(\mathbb{B})} \min_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \int_{\mathbb{A}} \int_{\mathbb{B}} c(a, b) \pi^{\mathbb{B}}(db) \pi^{\mathbb{A}}(da). \quad (\text{A.11})$$

**Remark A.22.** Theorem A.21 is the corollary from Theorem 3.3 as well as, in the case of a metric space  $\mathbb{A}$ , Proposition I.2.2 from Mertens et al. [17, p. 18] and minimax theorems from Mertens [16], because there is no assumption that  $\mathbb{A}$  is compact.

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